STABILITY OF EQUILIBRIUM

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  - Euler (statically indeterminate)
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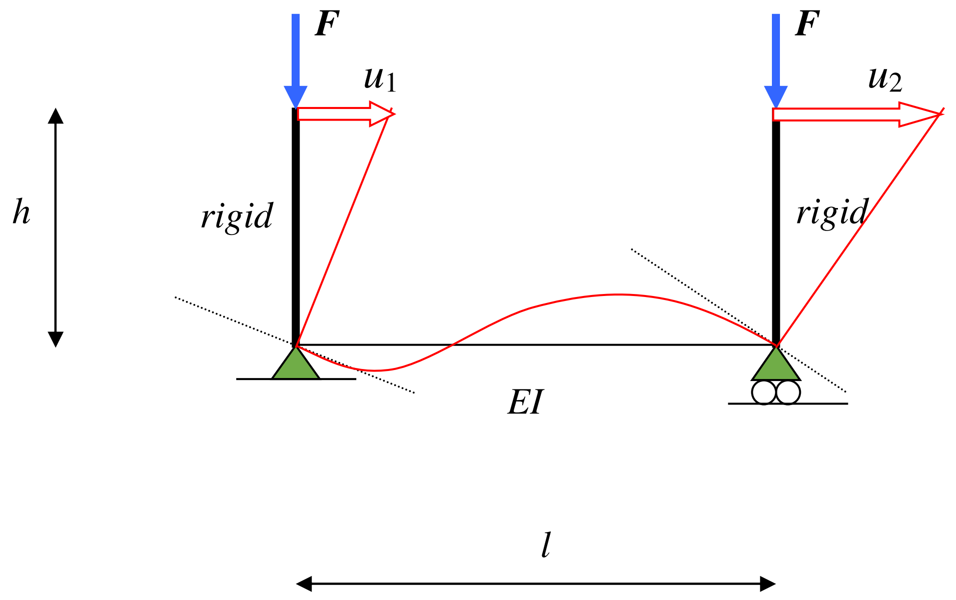
OBJECTIVES OF TODAY

- 3 Systems with two degrees of freedom
  - Derive the governing equations
  - Math recap
  - Find the governing critical load and the corresponding mode (shape) of the deformed structure
- 4 Systems with infinite degrees of freedom, Flexural Buckling
  - Euler (statically determinate)
  - Buckling shape, buckling force and buckling length
  - Examples using Euler
- 5 Buckling and the Engineering Code (Eurocode 3)
  - From Euler to the Engineering Code

STRUCTURES WITH 2 DEGREES OF FREEDOM

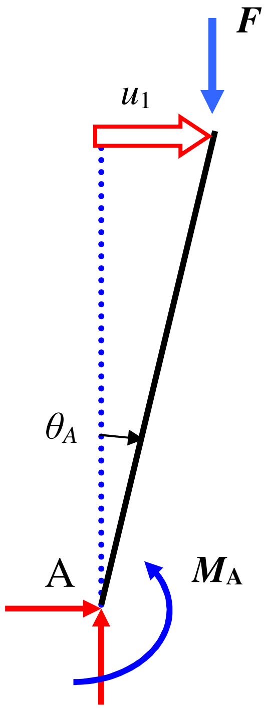


EQUILIBRIUM?

NOTE :

We do not make use of the fact that this structure is symmetric with a symmteric loading. We will reflect on this after obtaining our results.

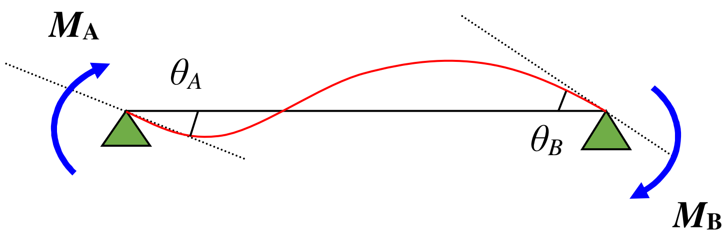
FREE BODY DIAGRAM - EQUILIBRIUM CONDITIONS -

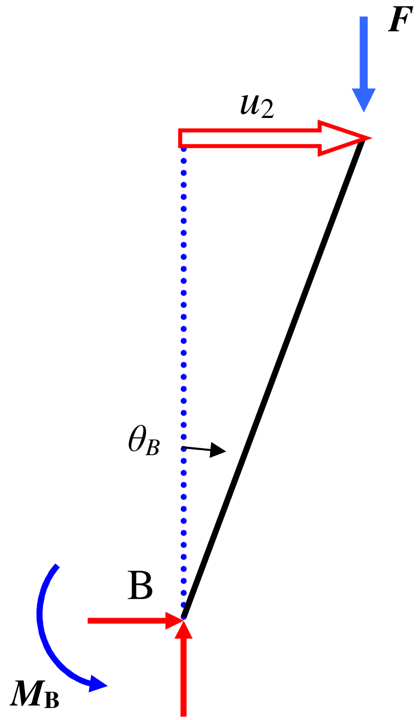


Released body

member A-B is a kind of spring:

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \frac{EI}{l} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$$





Released body

NOTE: 2 parts, so two equations

SUM OF MOMENTS FOR EACH PART

left part:

$$F\,h\,\theta_A = M_A \quad \Rightarrow \quad \left(\frac{4EI}{l} - F\,h\right)\theta_A + \frac{2EI}{l}\theta_B = 0$$

right part:

$$F\,h\,\theta_B = M_B \quad \Rightarrow \quad \frac{2EI}{l}\theta_A + \left(\frac{4EI}{l} - F\,h\right)\theta_B = 0$$

Do we recognise this system of equations?

ONE EXTRA STEP ...

$$\left(\frac{4EI}{l} - F\,h\right)\theta_A + \frac{2EI}{l}\theta_B = 0$$
$$\frac{2EI}{l}\theta_A + \left(\frac{4EI}{l} - F\,h\right)\theta_B = 0$$

$$= \begin{bmatrix} \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} - Fh \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4EI}{hl} & \frac{2EI}{hl} \\ \frac{2EI}{hl} & \frac{4EI}{hl} \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} - F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow [K]\bar{\theta} - F[I]\bar{\theta} = 0$$

matrix — unity matrix

EIGENVALUE PROBLEM

Math-recap (notation for 2D example in *x-y* plane)

$$\begin{bmatrix} k_{xx}-\lambda & k_{xy} \\ k_{yx} & k_{yy}-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=0 \;\Rightarrow\; \left(K-\lambda I\right)\underline{u}=0 \;\Rightarrow\; K\,\underline{u}=\lambda\,\underline{u}$$

- homogeneous system of equations (righthandside is equal to zero)
- only a non-trivial solution if the determinant of the matrix is zero.
- Solve the characteristic polynomial to find the roots for which the determinant becomes zero. We will find two values for λ
- These values are called the *eigenvalues*
- For each *eigenvalue* an *eigenvector* (mode) can be found
- To find this *eigenvector*, substitute the *eigenvalue* in the system of equations

APPLY THIS ON OUR EXAMPLE

(EIGENVALUE = CRITICAL LOAD)

$$define: \quad \beta = \frac{EI}{hl}$$

$$\begin{bmatrix} 4\beta - F & 2\beta \\ 2\beta & 4\beta - F \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} = 0$$

$$\det \begin{vmatrix} 4\beta - F & 2\beta \\ 2\beta & 4\beta - F \end{vmatrix} = 12\beta^2 - 8\beta F + F^2 = 0$$

$$(F - 2\beta)(F - 6\beta) = 0 \quad \Rightarrow \quad \boxed{F_1 = 2\beta} \quad ; \quad F_2 = 6\beta$$

lowest eigenvalue is governing

FIND THE EIGENVECTORS (EIGENVECTOR = CRITICAL MODE SHAPE)

Substitute per “mode” the eigenvalue in the system of equations:

$$F_1 = 2\beta$$

$$2\beta\theta_A + 2\beta\theta_B = 0$$

$$2\beta\theta_A + 2\beta\theta_B = 0 \qquad assume: \quad \theta_A = \mu \rightarrow \quad \theta_B = -\mu \Leftrightarrow \overline{\theta} = \mu \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

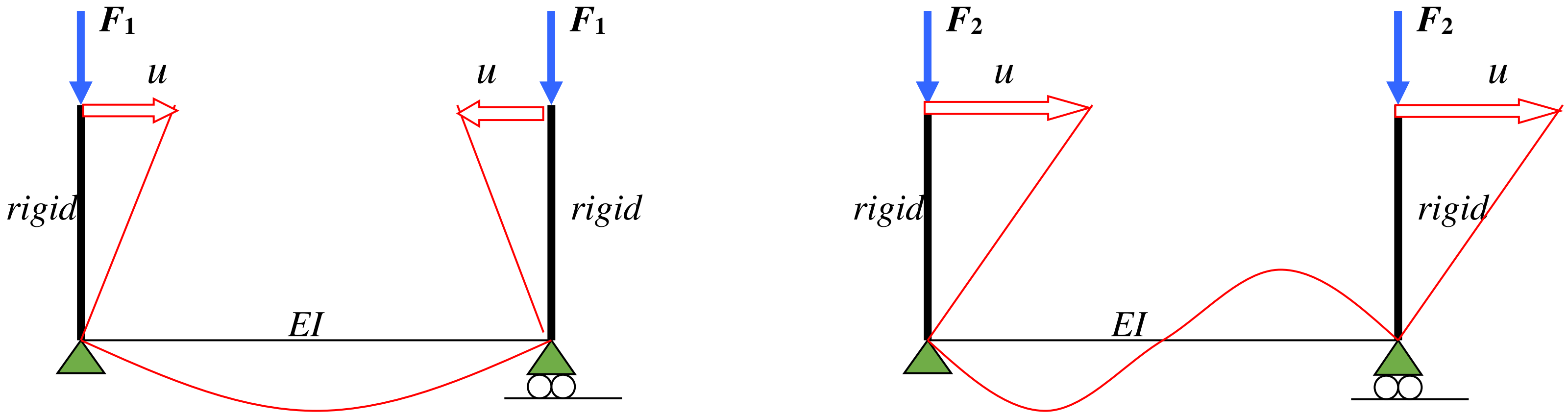
$$F_2 = 6\beta$$

$$-2\beta\theta_A + 2\beta\theta_B = 0$$

$$2\beta\theta_A - 2\beta\theta_B = 0 \qquad assume: \quad \theta_A = \lambda \rightarrow \quad \theta_B = \lambda \Leftrightarrow \overline{\theta} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

RESULT

(2 DEGREES OF FREEDOM = 2 EIGENVALUES = 2 EIGENVECTORS)



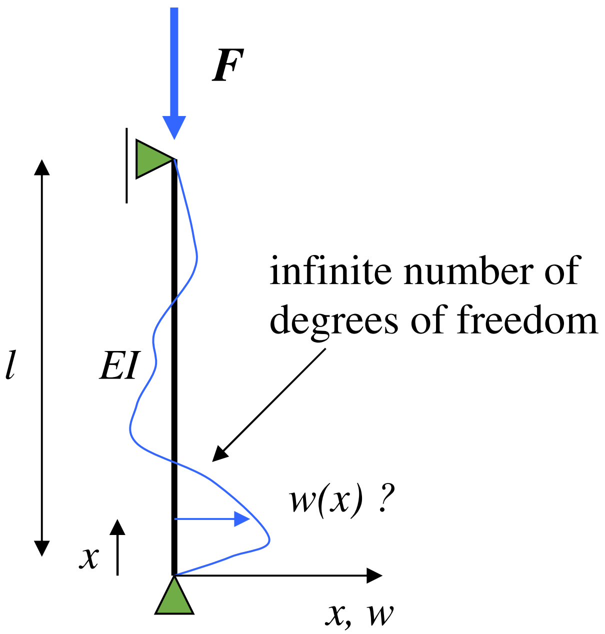
Lowest value = governing

$$F_k = \frac{2EI}{h\,l}$$

NOTE:

The mode shape is defined, the magnitude of the displacement is NOT!

STRUCTURES WITH INFINITE DEGREES OF FREEDOM, FLEXURAL BUCKLING – basic case -



Leonhard Euler

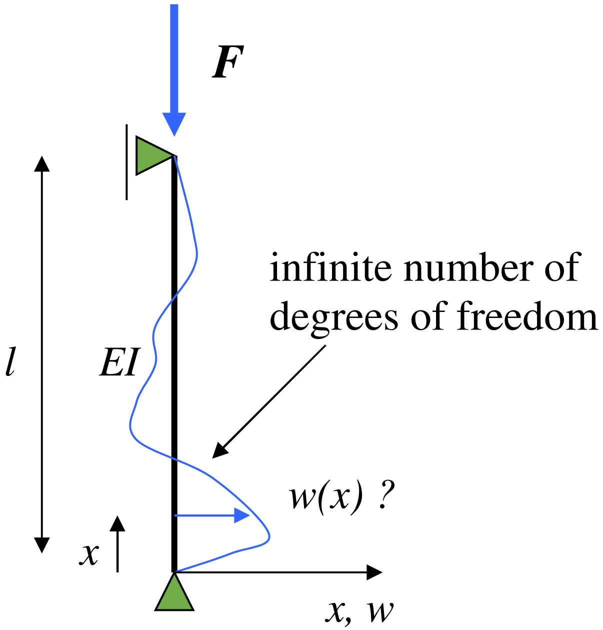
Portrait by Jakob Emanuel Handmann, 1753

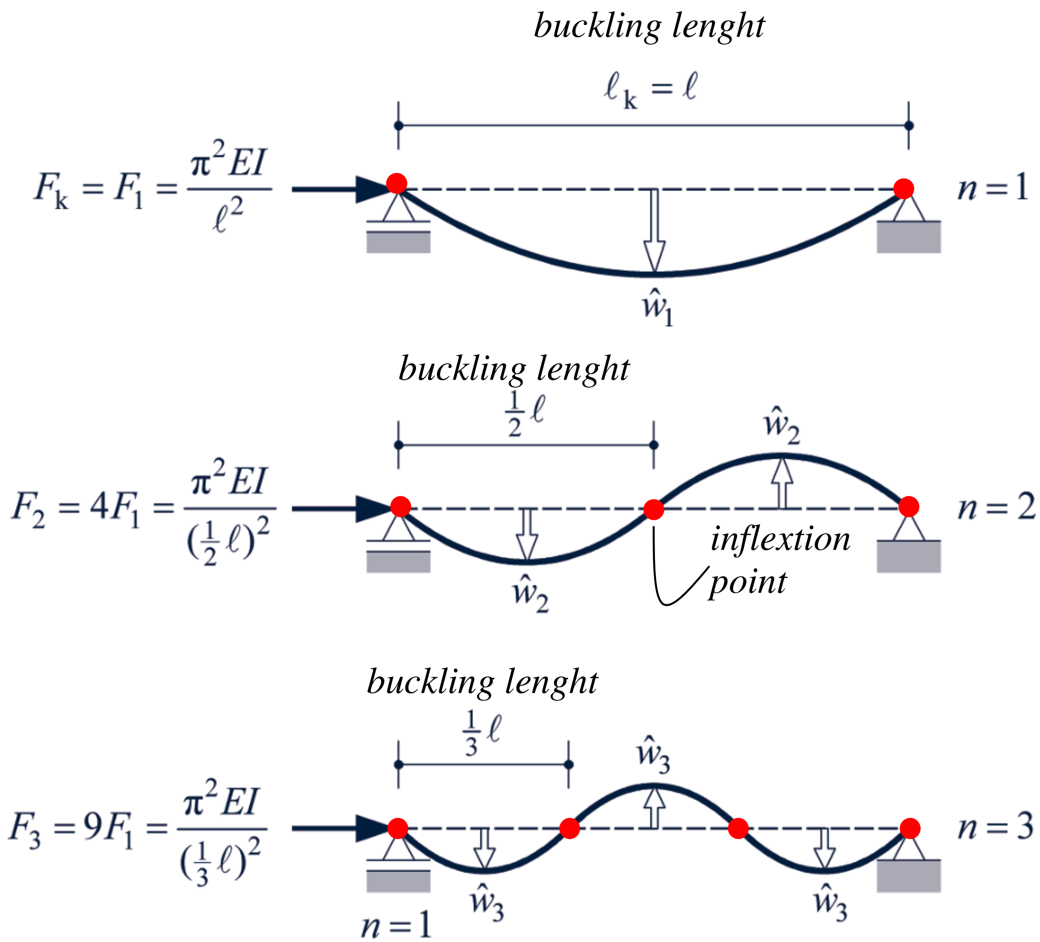
Born	15 April 1707 Basel, Swiss Confederacy
Died	18 September 1783 (aged 76) [OS: 7 September 1783] Saint Petersburg, Russian Empire

DERIVE THE GOVERNING EQUATION TO FIND:

- THE CRITICAL LOAD AND
- THE CORRESPONDING MODE

RESULT



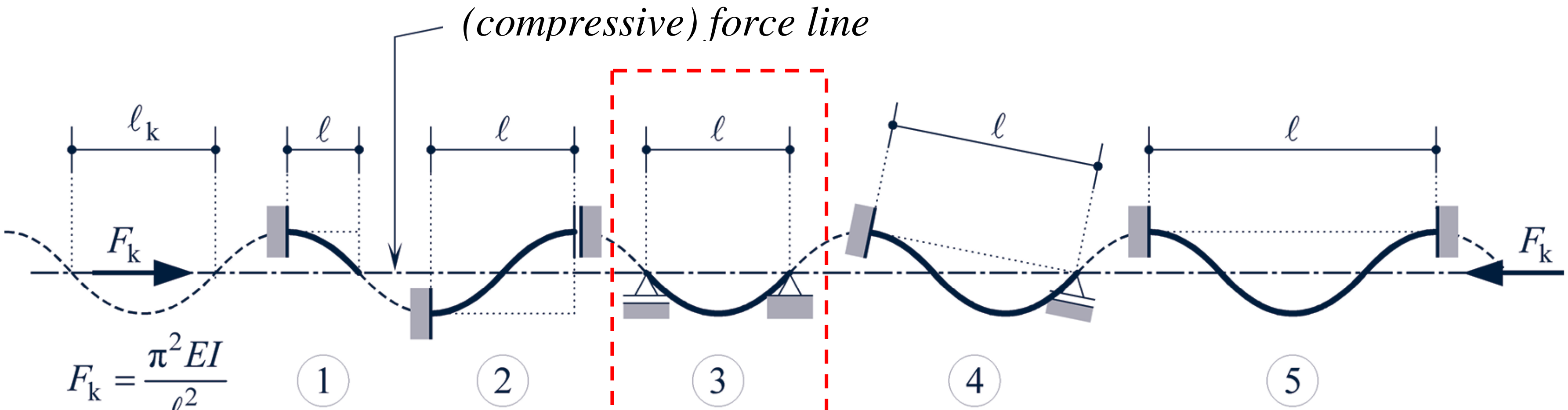


EULER

$$F_k = \frac{\pi^2 EI}{l_k^2}$$

critical load → buckling mode (shape) → buckling length

STANDARD SOLUTIONS FOR BUCKLING (EULER)



$$F_k = \frac{\pi^2 EI}{\ell_k^2}$$

1: $\begin{cases} \ell_k = 2\ell \\ F_k = \dfrac{\pi^2 EI}{4\ell^2} \end{cases}$

2: $\begin{cases} \ell_k = \ell \\ F_k = \dfrac{\pi^2 EI}{\ell^2} \end{cases}$

3: $\begin{cases} \ell_k = \ell \\ F_k = \dfrac{\pi^2 EI}{\ell^2} \end{cases}$

4: $\begin{cases} \ell_k \approx 0,7\ell \\ F_k \approx \dfrac{2\pi^2 EI}{\ell^2} \end{cases}$

5: $\begin{cases} \ell_k = 0,5\ell \\ F_k = \dfrac{4\pi^2 EI}{\ell^2} \end{cases}$

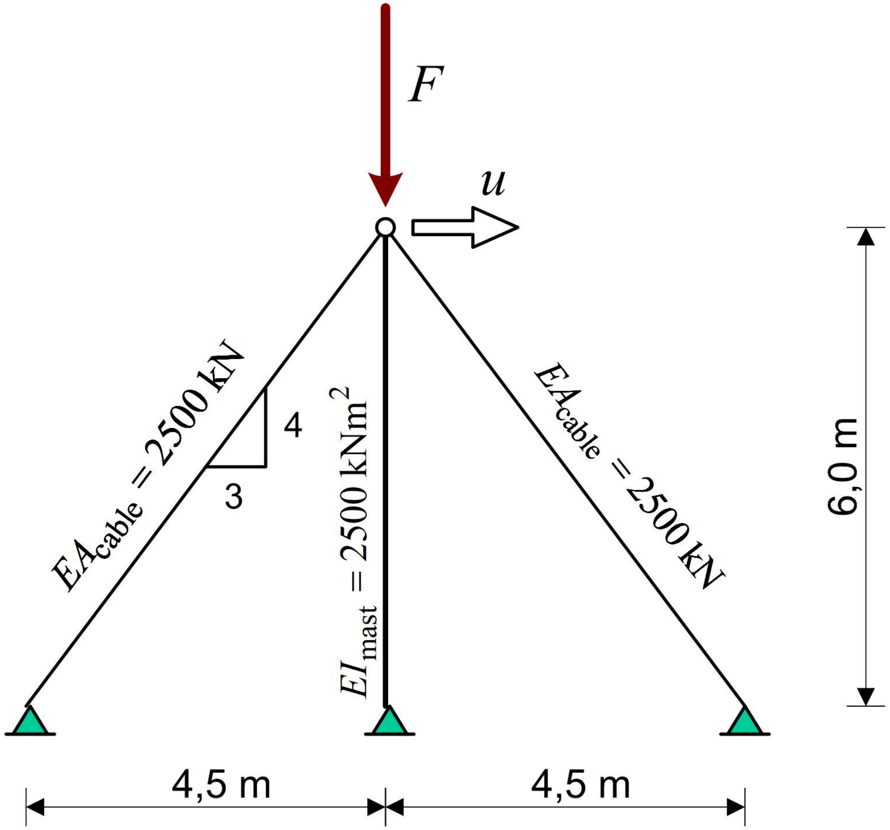
SWAY models (1, 2) **NON-SWAY models** (3, 4, 5)

Use assignment to become familiar with these standard cases!

EXAMPLE

$assume:$

$u \ll 1$



Question:

Find the critical load F_c

BUCKLING AND THE CODE (EUROCODE 3)

- VERIFY THE COMPRESSIVE FORCE IN THE ULTIMATE LIMIT STATE (FAILURE !!!)

- BUCKLING IS A (DANGEROUS = SAFETY ISSUE = = FAILURE) PHENOMENON WHICH MUST BE TESTED IN THE ULTIMATE LIMIT STATE

- HOW TO GET FROM EULER (BUCKLING FORCE) TO THE VERIFICATION OF DESIGN VALUES FOR COMPRESSIVE FORCES ?

BUCKLING STRESS

Euler $$F_{\text{Euler}} = \frac{\pi^2 EI}{L_{\text{cr}}^2}$$

Stress $$\sigma_{\text{b}} = \frac{F_{\text{Euler}}}{A} \quad (buckling)$$

Buckling stress $$\sigma_{\text{b}} = \pi^2 E \frac{I}{L_{\text{cr}}^2 A} \quad \text{with: } i = \sqrt{\frac{I}{A}} \; (radii\ of\ inertia)$$

Buckling stress $$\sigma_{\text{b}} = \pi^2 E \frac{i^2}{L_{\text{cr}}^2} = \frac{\pi^2 E}{\lambda^2} \quad \text{with: } \lambda = \frac{L_{\text{cr}}}{i} \; (slenderness)$$

BUCKLING STRESS "AS A FUNCTION"

$$\sigma_b \le f_y$$

$$\frac{\pi^2 E}{\lambda^2} \le f_y$$

LIMIT VALUE:

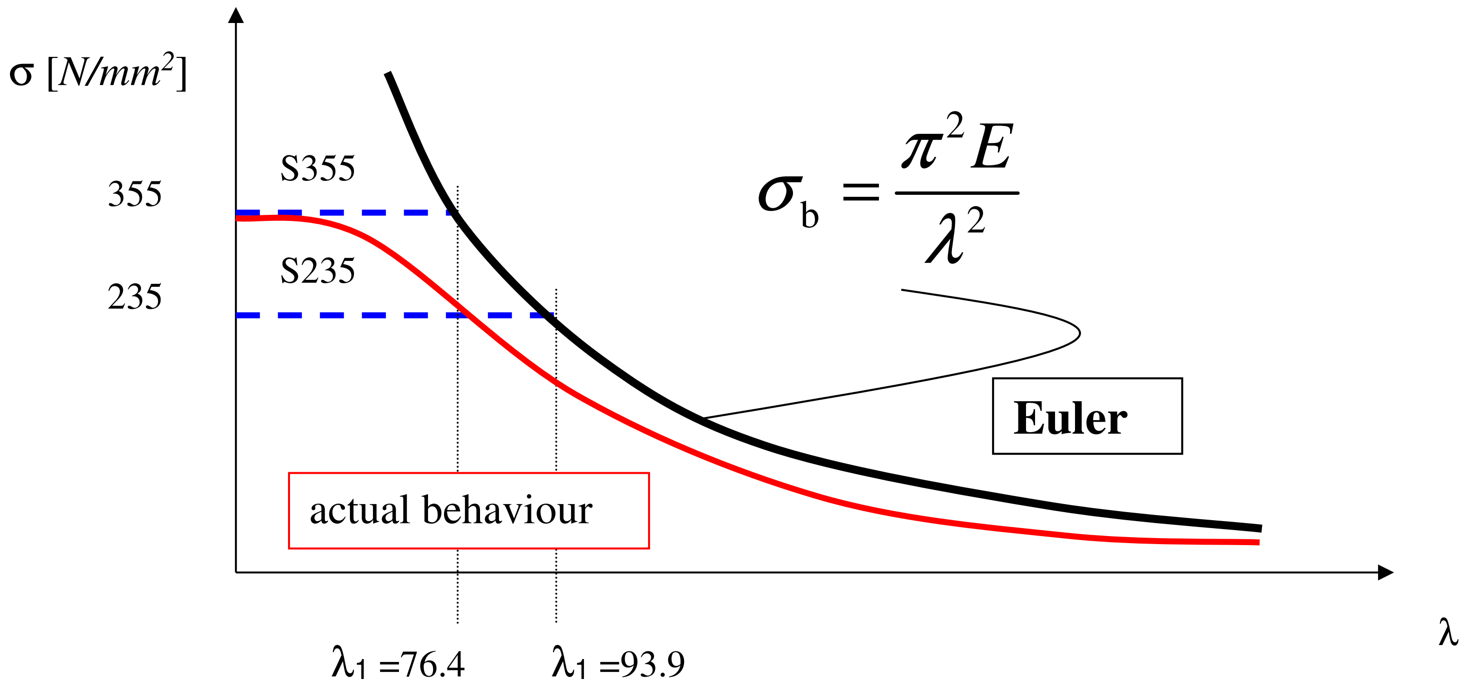
$$\lambda_1 = \sqrt{\frac{\pi^2 E}{f_y}}$$

STEEL E=2.1×10⁵ N/mm²

For different types of steel

S355	yielding	f_y= 355 N/mm²	λ_1=76,4
S235	yielding	f_y = 235 N/mm²	λ_1=93,9

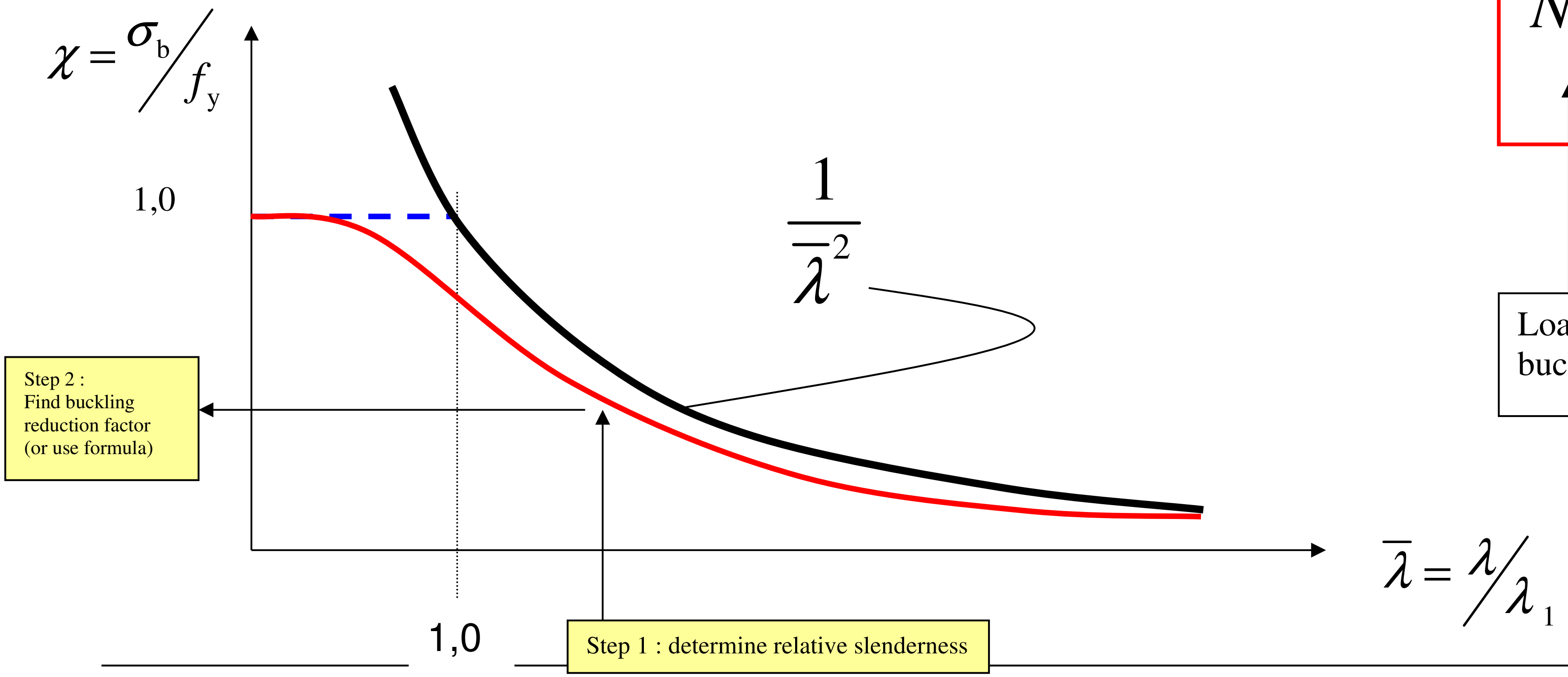
STRESS AS A "BUCKLING CURVE"



BUCKLING CURVE (CODE)

Dimensionless by substitution of:

$$\begin{cases} y-axis & \chi = \dfrac{\sigma_k}{f_y} \\ x-axis & \bar{\lambda} = \dfrac{\lambda}{\lambda_1} \end{cases}$$



Step 3 : find load bearing capacity

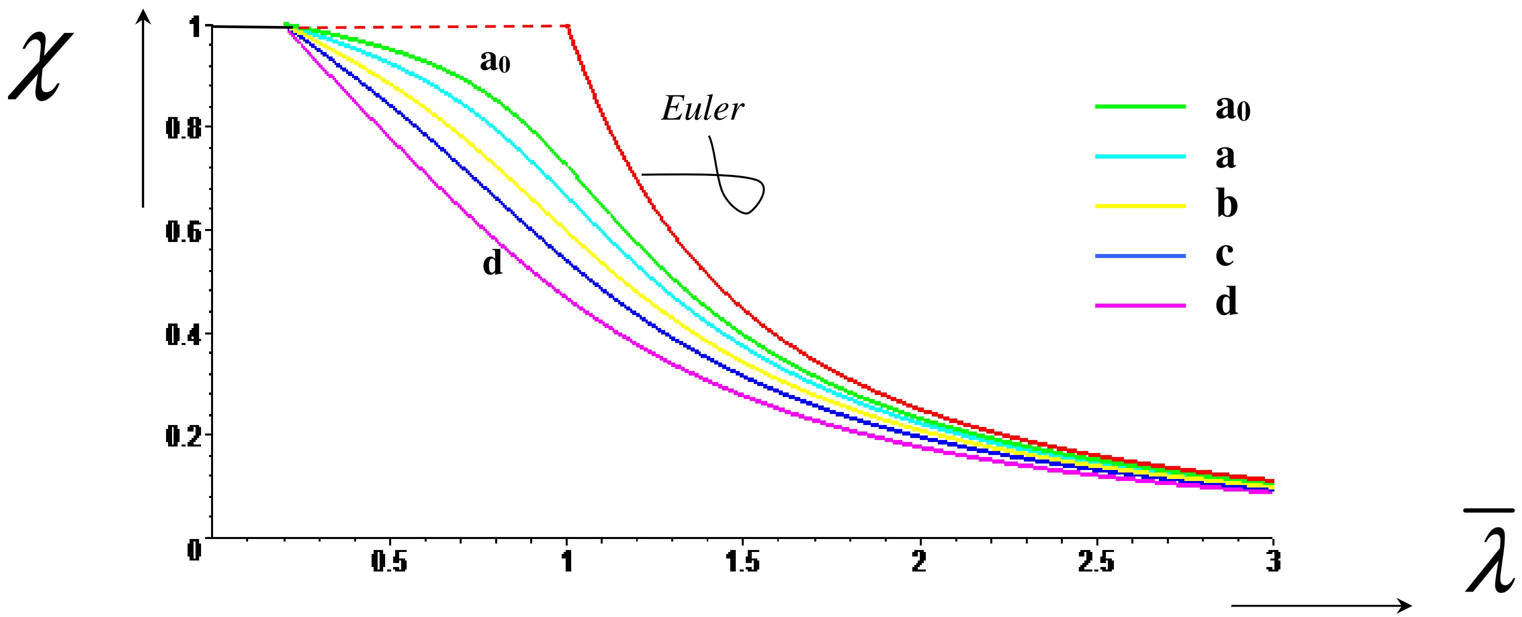
$$\sigma_b = \chi \cdot f_y$$

$$N_{b,Rd} = \frac{\chi \cdot A \cdot f_y}{\gamma_{M1}}$$

Load bearing capacity of the buckling element

EUROCODE 3

Buckling curve depends on the type of cross section



Design value for the normal force (loading)

$$N_{Ed} < N_{b,Rd}$$

Design value for the capacity (strength)