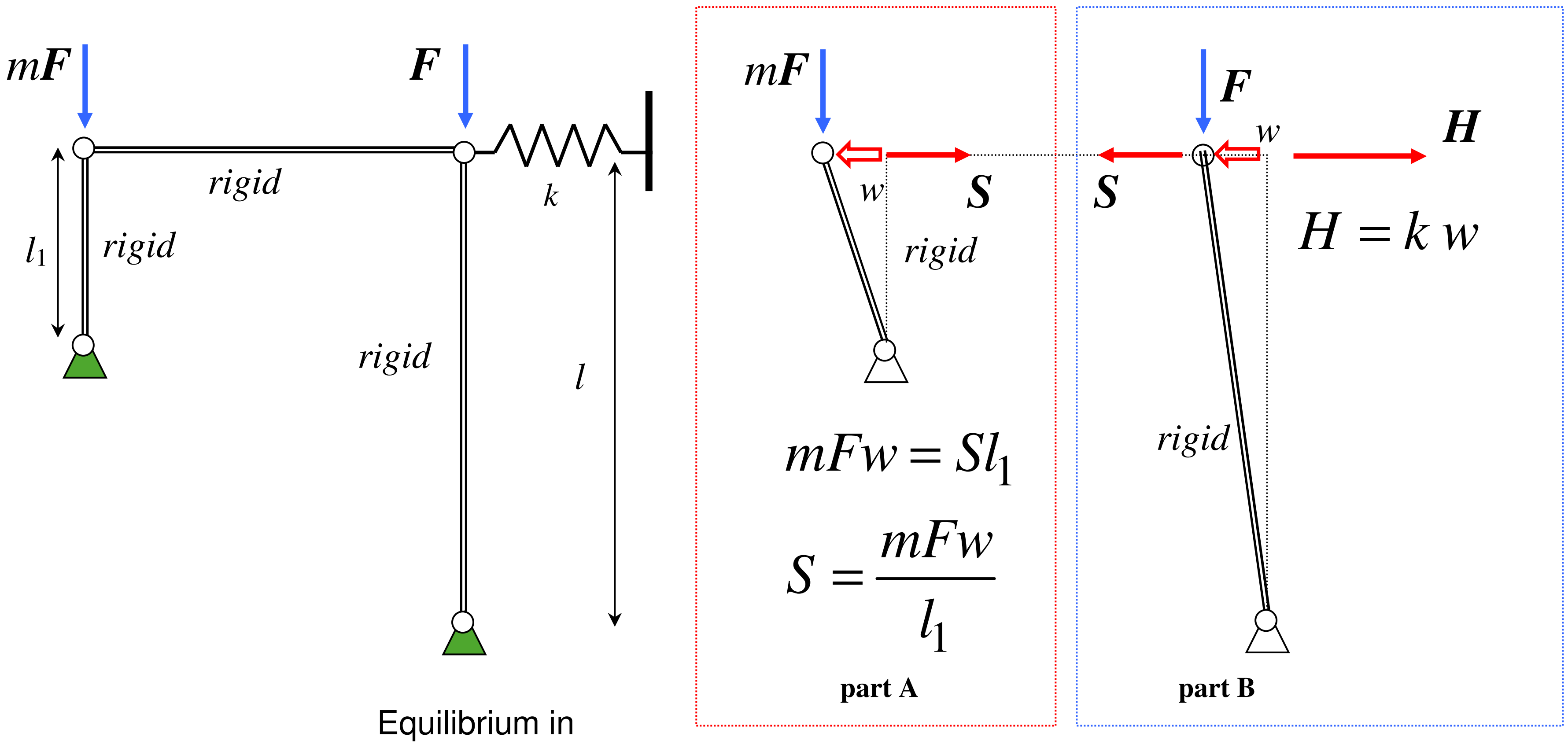
OBJECTIVES OF THIS PART

- 4 Systems with infinite degrees of freedom, Flexural Buckling
  - Euler (statically determinate)
  - Buckling shape, buckling force and buckling length
  - Examples using Euler
  - Euler (statically indeterminate)
  - Basic solutions of Euler
  - Flexible supported beam in compression, sway and non-sway
  - **Coupled systems, effective load on stability element**

- Examine structural models with vertical loading on different elements which are coupled.
- Model the load transfer toward the stabilizing element
- Perform small research to create an engineering (design) formula

COUPLED SYSTEMS, EFFECTIVE LOAD

(rigid model with two-force elements and a spring)



Equilibrium in displaced position, part A

EQUILIBRIUM IN DISPLACED POSITION, part B

$$F w + S\,l - H\,l = 0 \qquad \text{with: } S = \frac{mFw}{l_1}$$

$$F\,w + \frac{mF}{l_1}\,l\,w = k\,l\,w$$

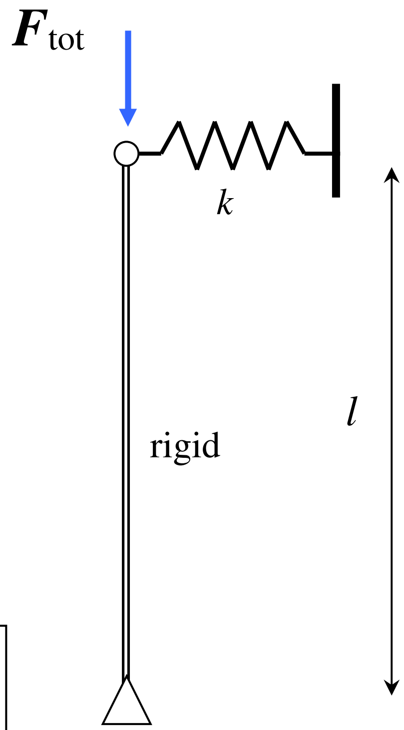
$$F + \frac{mF}{l_1}\,l = k\,l$$

Influence of load on non-stabilized element

EQUIVALENT MODEL

$$F_k = k\,l \quad \Leftrightarrow$$

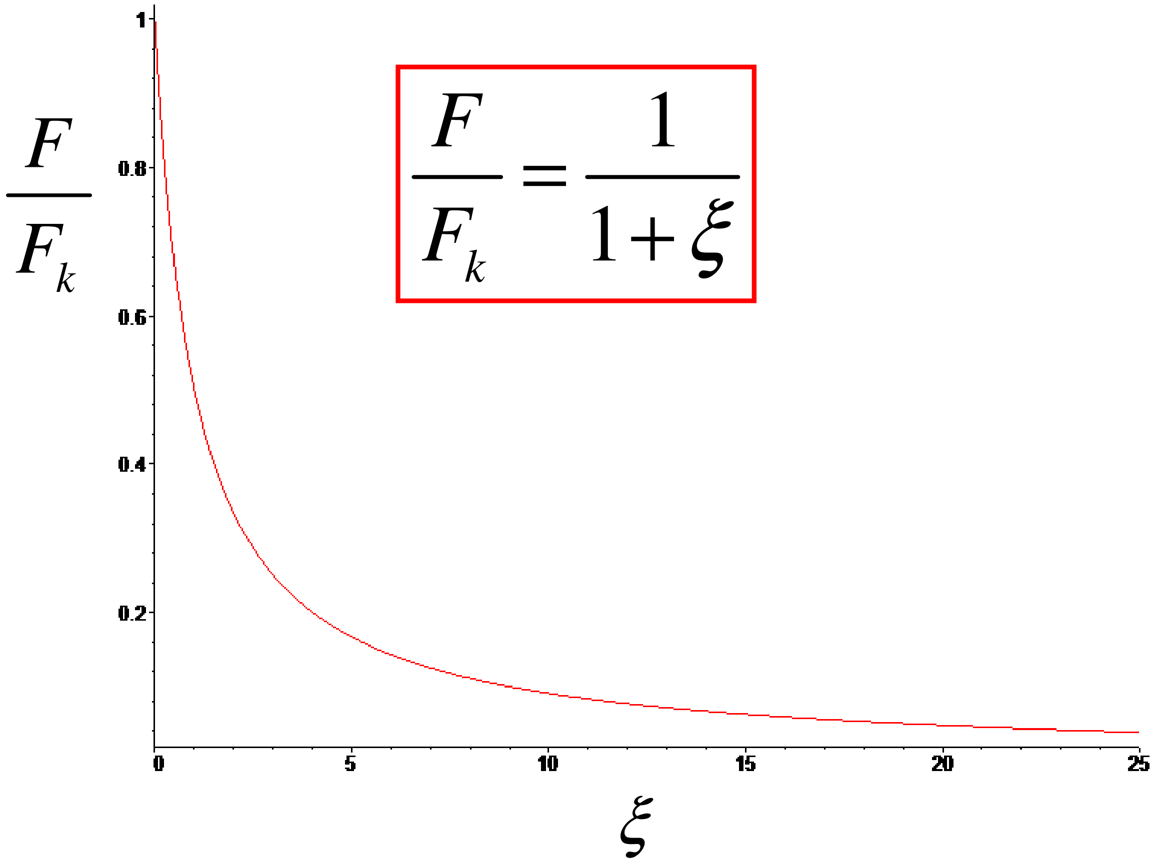
$$F + \frac{mF}{l_1} l = F_{\text{tot}} \leq F_k \quad \text{or} \quad \frac{F}{F_k} = \frac{1}{1 + \dfrac{ml}{l_1}}$$



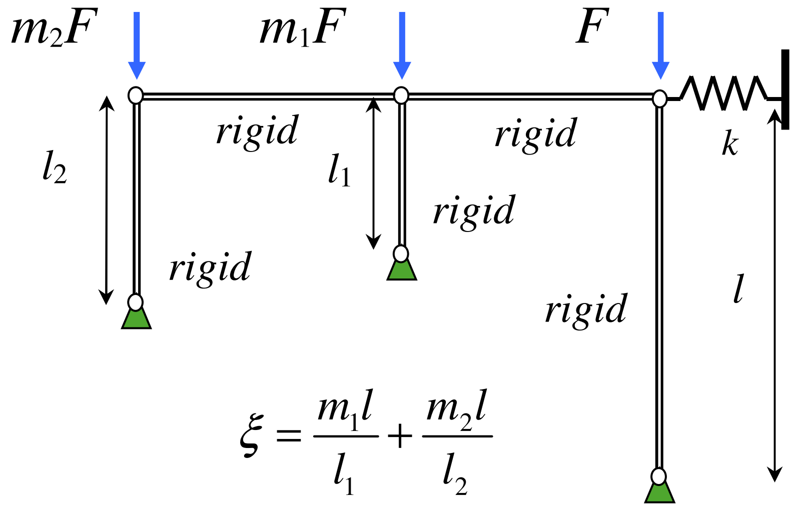
The effective load on the stabilizing element is the load on this element itself plus the effect of the load on the non-stabilized elements. These loads have to be multiplied with the length of the stabilized element and divided by the element it acts on. This acts like a kind of penalty l/l_1.

Only if all elements have the same length, the total load on the stabilizing element is the sum of all loads.

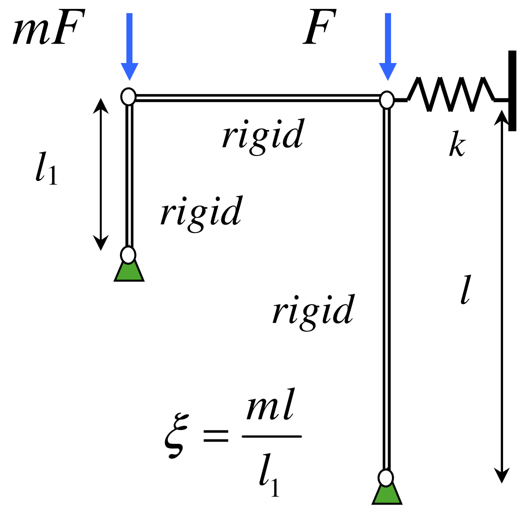
CRITICAL LOAD



$$\frac{F}{F_k} = \frac{1}{1+\xi}$$



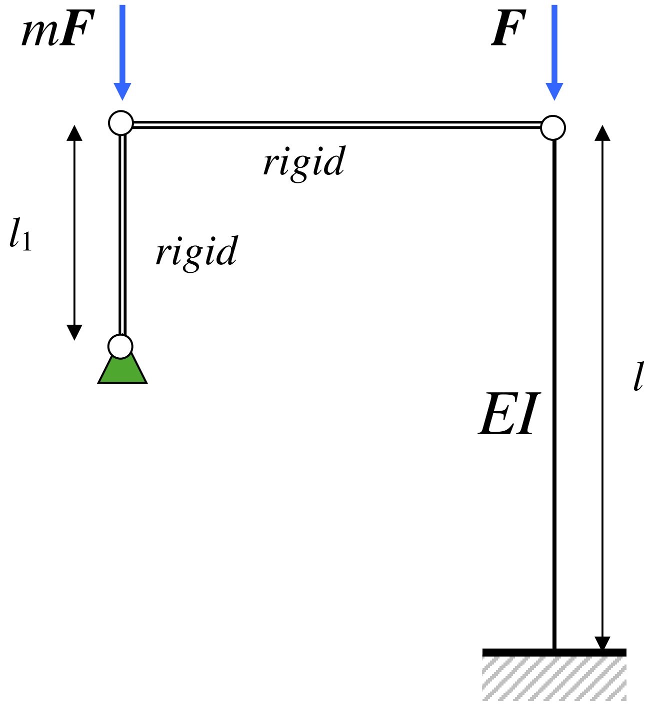
$$\xi = \frac{m_1 l}{l_1} + \frac{m_2 l}{l_2}$$



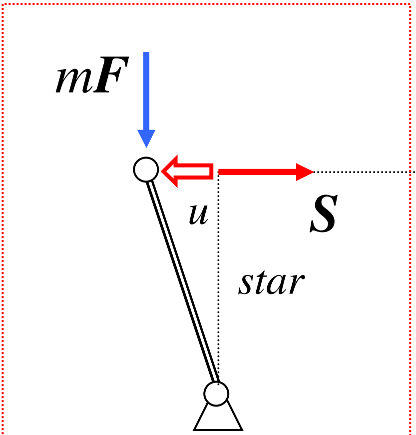
$$\xi = \frac{ml}{l_1}$$

COUPLED SYSTEMS, EFFECTIVE LOAD

(flexural model with two-force elements and a spring)

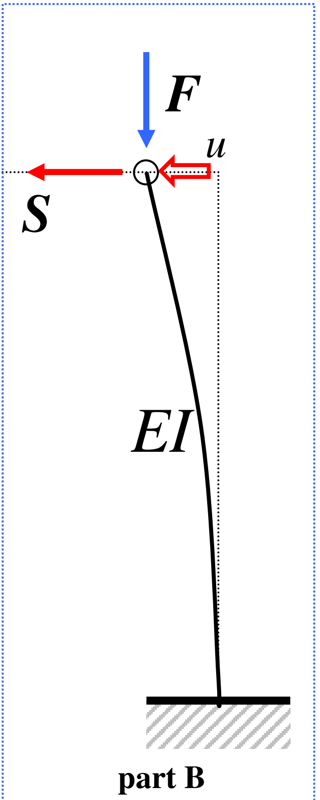


Equilibrium in displaced position, part A

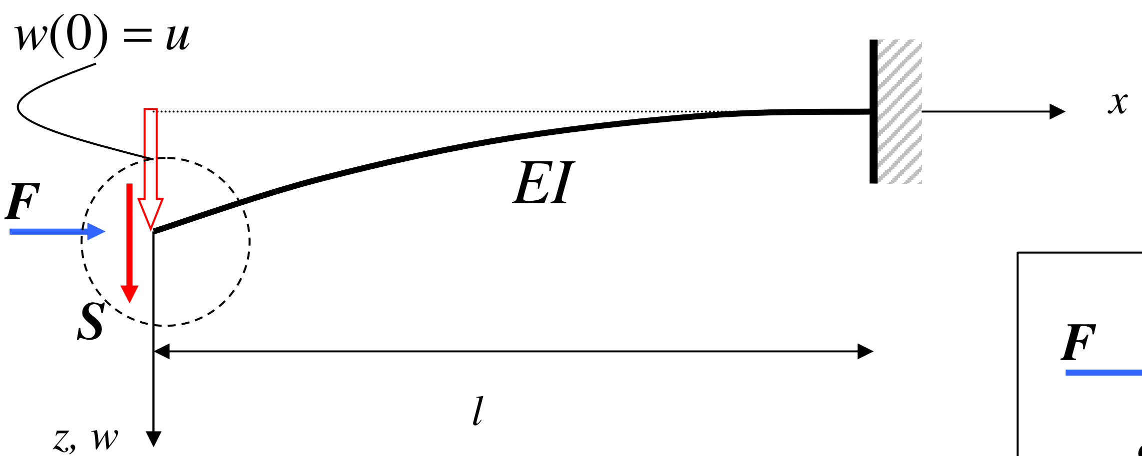


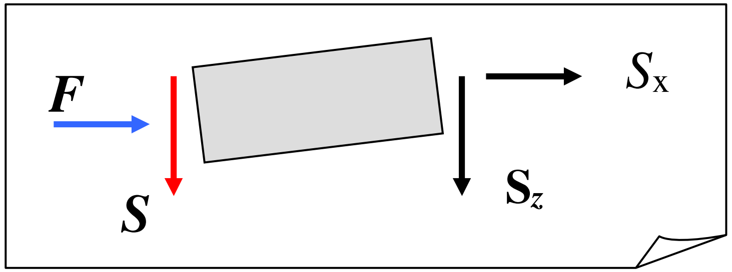
$$mFu = Sl_1$$

$$S = \frac{mFu}{l_1}$$



EQUILIBRIUM PART B





$$w'''' + \alpha^2 w'' = 0 \quad \text{with:} \quad \alpha^2 = \frac{F}{EI}$$

$$S_z = M' - Fw' \Leftrightarrow S_z = -EIw''' - Fw'$$

Boundary conditions:

1) $S_z(0) = -S$ 3) $w(l) = 0$

2) $M(0) = 0$ 4) $\varphi(l) = 0$

SOLVE

slope of the force line

tooling:

$$w(x) = C_1 + C_2 x + C_3 \cos\alpha x + C_4 \sin\alpha x$$

$$w'(x) = C_2 - \alpha C_3 \sin\alpha x + \alpha C_4 \cos\alpha x \qquad S_z = -EIw''' - Fw' = -FC_2$$

$$w''(x) = -\alpha^2 C_3 \cos\alpha x - \alpha^2 C_4 \sin\alpha x$$

Boundary conditions:

Use:

$$S = \frac{mF.w(0)}{l_1}$$

1) $-FC_2 = -\frac{mF}{l_1}(C_1 + C_3)$

2) $EI\alpha^2 C_3 = 0 \quad \Rightarrow \quad C_3 = 0$

3) $C_1 + C_2 l + C_3 \cos\alpha l + C_4 \sin\alpha l = 0$

4) $C_2 - \alpha C_3 \sin\alpha l + \alpha C_4 \cos\alpha l = 0$

RESULT:

$$\begin{bmatrix} m & -l_1 & 0 \\ 1 & l & \sin\alpha l \\ 0 & 1 & \alpha\cos\alpha l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Homogeneous set of equations. Only a non-trivial solution possible if the determinant is equal to zero:

$$\tan\alpha l = \frac{1}{m}\left(\frac{l_1}{l} + m\right)\alpha l \quad \wedge \quad \alpha^2 = \frac{F}{EI}$$

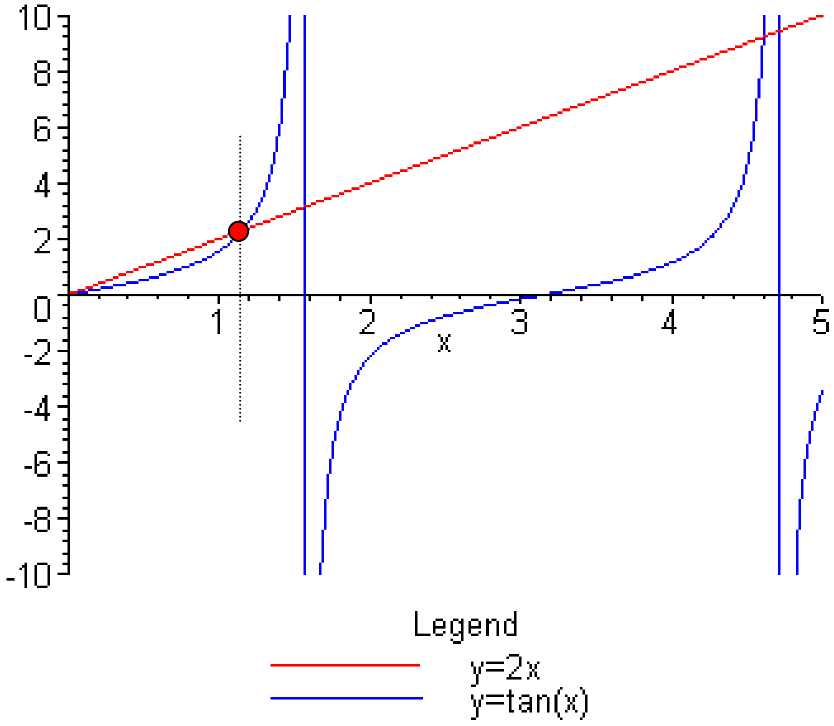
TRANSCENDENTAL EQUATION → MAPLE

SOLUTION for $l_1 = l$ and $m = 1$

$$\alpha l \cong 1{,}1655 \cong \frac{7}{6}$$

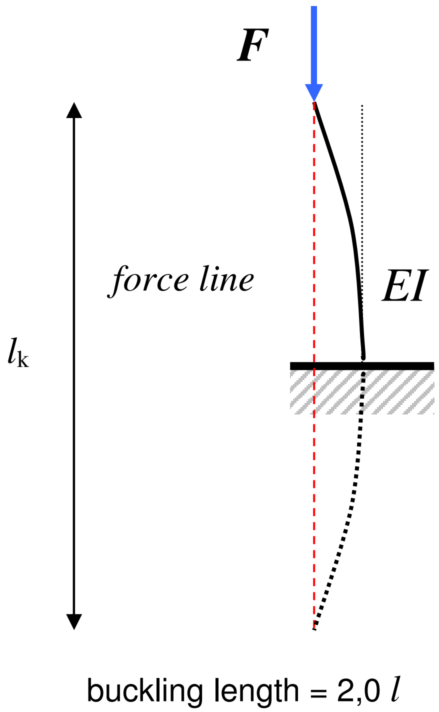
$$F_k = \frac{EI}{\left(\frac{6}{7}l\right)^2} = \frac{\pi^2 EI}{\left(\frac{6\pi}{7}l\right)^2}$$

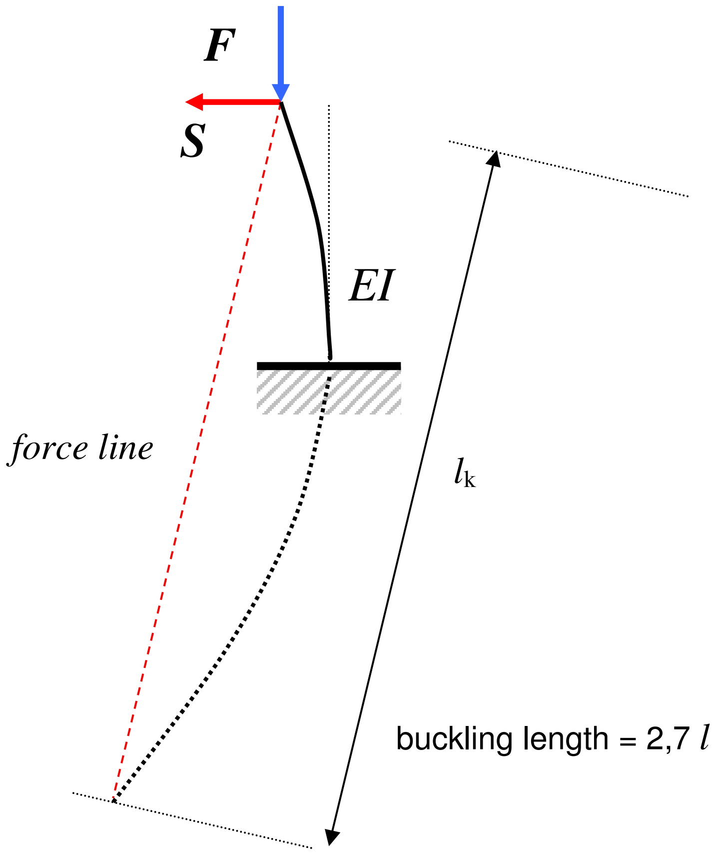
$$l_k = \tfrac{6\pi}{7} l \cong 2{,}7 l$$



RESULT

buckling length increases beyond $2l$

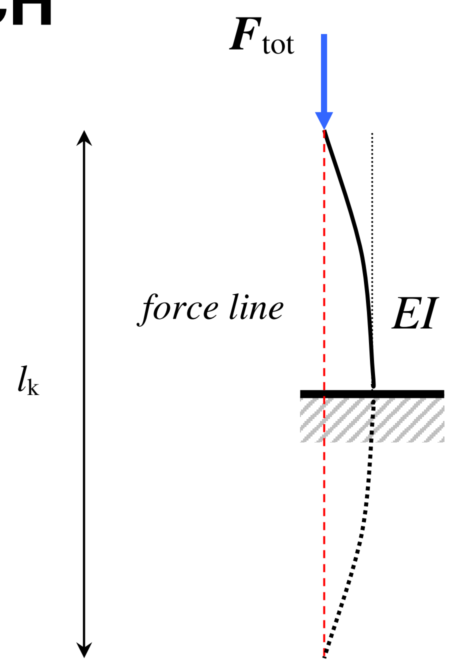




SUGGESTION FOR SIMPLIFIED APPROACH
allowed?

$$exact:$$

$$F_k = \frac{\pi^2 EI}{(2,69l)^2} = \frac{\pi^2 EI}{1,8\times(2l)^2} = 0,55\times\frac{\pi^2 EI}{(2l)^2}$$



buckling length = 2,0 l

$$basic\ Euler:$$

Put all load on stabilizing element in case columns have the same length.

$$2F_k = \frac{\pi^2 EI}{(2l)^2}$$

$$F_k = \frac{\pi^2 EI}{2,0(2l)^2} = 0,5\times\frac{\pi^2 EI}{(2l)^2}$$

Not exact but conservative result which is 10% off.

INFLUENCE OF THE LENGTH TWO-FORCE ELEMENT $(m = 1)$

for each $\frac{l_1}{l}$ an αl can be found:

$$F_k = EI\alpha^2 = \frac{EI}{l^2} \times (\alpha l)^2 = \frac{\pi^2 EI}{l^2} \times \left(\frac{\alpha l}{\pi}\right)^2 = \frac{\pi^2 EI}{(\beta l)^2}$$

$$\gamma = \frac{F_k}{F_E} = \frac{\dfrac{\pi^2 EI}{(\beta l)^2}}{\dfrac{\pi^2 EI}{4l^2}} = \frac{4}{\beta^2} \qquad \text{with: } \beta = \frac{\pi}{\alpha l} \qquad \text{and: } l_k = \beta l$$

l_1/l	αl	β	γ
0,050	0,3765	8,34	0,06
0,125	0,5710	5,50	0,13
0,250	0,7595	4,14	0,23
0,750	1,0867	2,89	0,48
1,000	1,1667	2,69	0,55
2,000	1,3242	2,37	0,71
3,000	1,3933	2,25	0,79
4,000	1,4320	2,19	0,83
5,000	1,4569	2,16	0,86
10,000	1,5107	2,08	0,92
15,000	1,5300	2,05	0,95
1000,00	1,5708	2,00	1,00

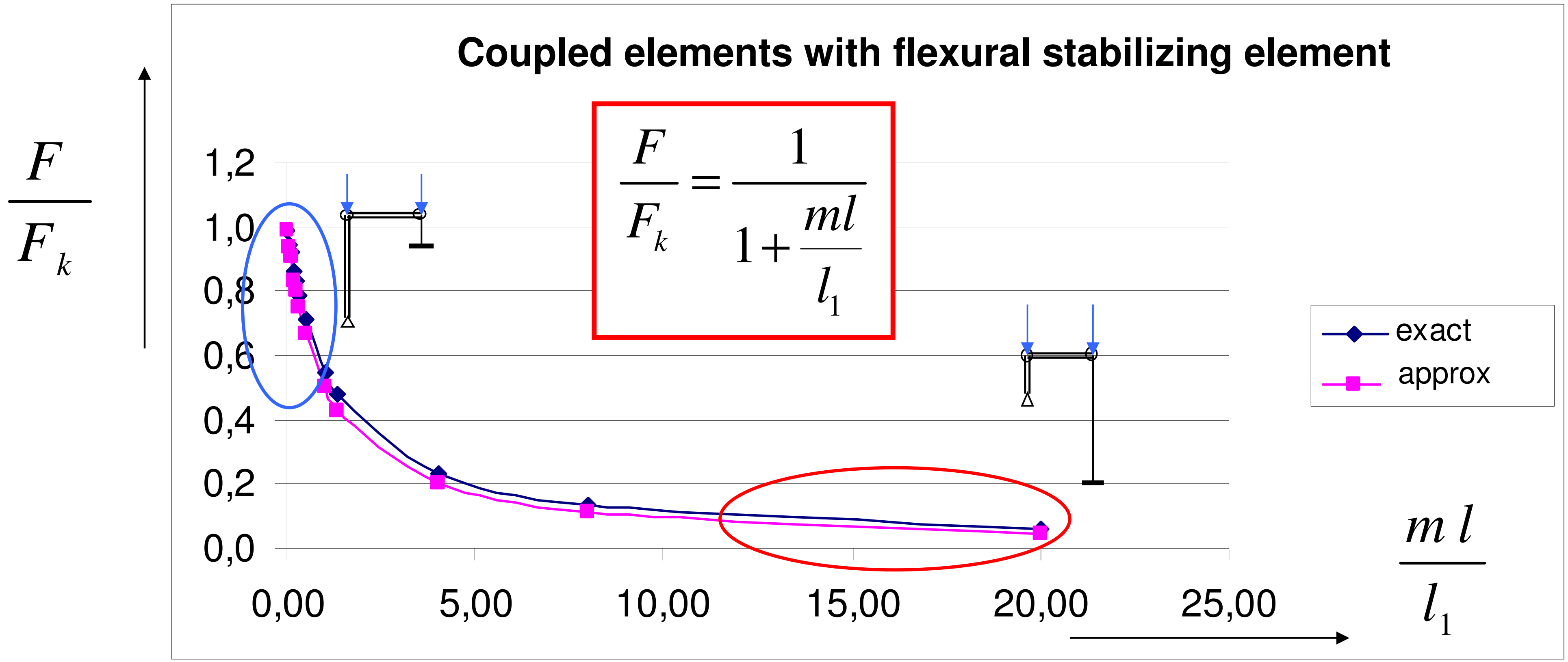
example:

$$l_1 = 0{,}125 l \quad \Rightarrow \quad \gamma = 0{,}13$$

short two-force element reduces the buckling load!

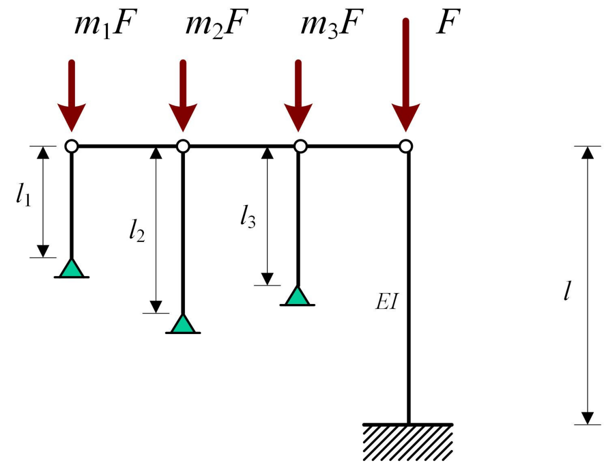
INFLUENCE OF THE LENGTH OF THE TWO-FORCE ELEMENT

(flexural model with two-force elements and a spring)



SO FAR PROMISING RESULT

Research needed for more basic situations to check the proposed design approach:



$$\frac{F}{F_k} = \frac{1}{1+\xi} \quad \text{with:} \quad \xi = \frac{m_1 l}{l_1} + \frac{m_2 l}{l_2} + \ldots$$

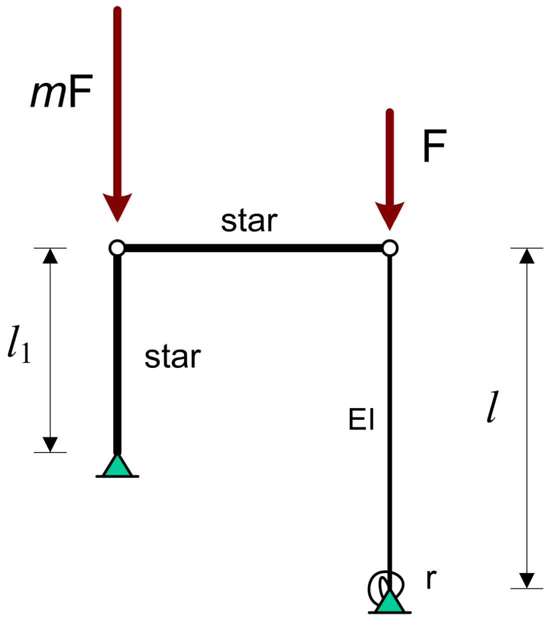
and:

$$F_k = \frac{\pi^2 EI}{4l^2}$$

As an example, we can check a situation with rotational spring.

EXAMPLE COUPLED SYSTEM

(flexural stabilizing element with a rotational spring)



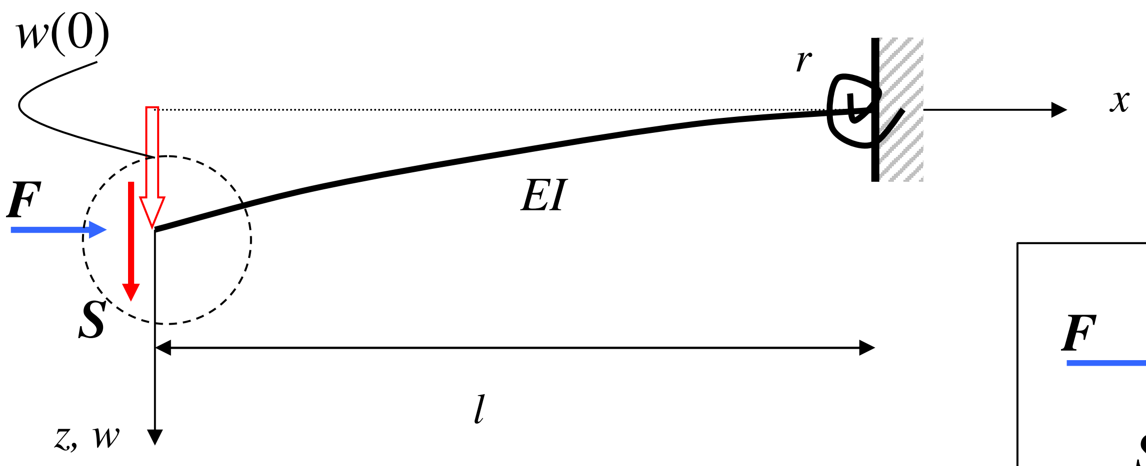
$$\frac{F}{F_k} = \frac{1}{1 + \dfrac{ml}{l_1}} \quad ??$$

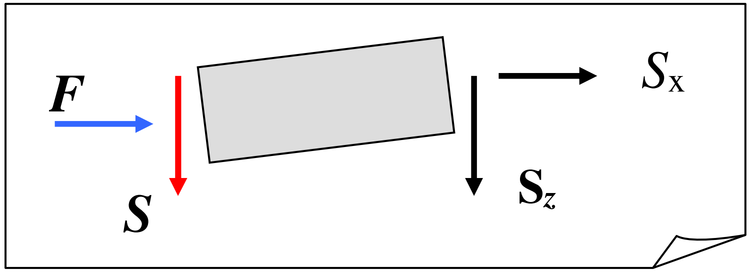
with:

$$F_k = \frac{1}{\eta}\frac{\pi^2 EI}{l^2} \quad \text{and } \eta = 4 + \frac{10}{\rho}; \; \rho = \frac{rl}{EI}$$

In order to compare we need the exact solution for this case …..

EQUILIBRIUM OF PART B





$$w'''' + \alpha^2 w'' = 0 \quad \text{with:} \quad \alpha^2 = \frac{F}{EI}$$

$$S_z = M' - Fw' \Leftrightarrow S_z = -EIw''' - Fw'$$

Boundary conditions:

1) $S_z(0) = -S$
3) $w(l) = 0$
2) $M(0) = 0$
4) $M(l) = r\varphi(l)$

SOLVE

slope of the force

Tooling:

$$w(x) = C_1 + C_2 x + C_3 \cos \alpha x + C_4 \sin \alpha x$$

$$w'(x) = C_2 - \alpha C_3 \sin \alpha x + \alpha C_4 \cos \alpha x \qquad S_z = -EIw''' - Fw' = -FC_2$$

$$w''(x) = -\alpha^2 C_3 \cos \alpha x - \alpha^2 C_4 \sin \alpha x$$

Boundary conditions:

Use :

$$S = \frac{mF.w(0)}{l_1}$$

$$1) \quad -FC_2 = -\frac{mF}{l_1}(C_1 + C_3)$$

$$2) \quad EI\alpha^2 C_3 = 0 \qquad \Rightarrow \qquad C_3 = 0$$

$$3) \quad C_1 + C_2 l + C_3 \cos \alpha l + C_4 \sin \alpha l = 0$$

$$4) \quad -rC_2 + (EI\alpha^2 \cos \alpha l + r\alpha \sin \alpha l)C_3 + (EI\alpha^2 \sin \alpha l - \alpha r \cos \alpha l)C_4 = 0$$

ELABORATE ...

$$\begin{bmatrix} m & -l_1 & 0 \\ 1 & l & \sin\alpha l \\ 0 & -r & EI\alpha^2\sin\alpha l - \alpha r\cos\alpha l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For infinite ρ this turns into the expression on page 13

This homogeneous system of equations has a non-trivial solution only if the determinant of the system is equal to zero:

$$\tan\alpha l = \frac{\left(\frac{l_1}{l} + m\right)\rho}{\alpha l\left(m + \frac{l_1}{l}\right) + \frac{m\rho}{\alpha l}} \quad \wedge \quad \alpha^2 = \frac{F}{EI} \quad \text{and } \rho = \frac{rl}{EI}$$

→ **MAPLE, Python**

Try this

EXACT RESULTS $(m = 2, \rho = 5)$

for each $\frac{l_1}{l}$ an αl can be found:

$$F = EI\alpha^2 = \frac{EI}{l^2} \times (\alpha l)^2 = \frac{\pi^2 EI}{l^2} \times \left(\frac{\alpha l}{\pi} \right)^2 = \frac{\pi^2 EI}{(\beta l)^2}$$

$$\gamma = \frac{F}{F_E} = \frac{\dfrac{\pi^2 EI}{(\beta l)^2}}{\dfrac{\pi^2 EI}{\eta l^2}} = \frac{\eta}{\beta^2}$$

with: $\beta = \frac{\pi}{\alpha l}$ and: $l_k = \beta l$

$\eta = 6$ with: $\rho = \frac{rl}{EI} = 5$

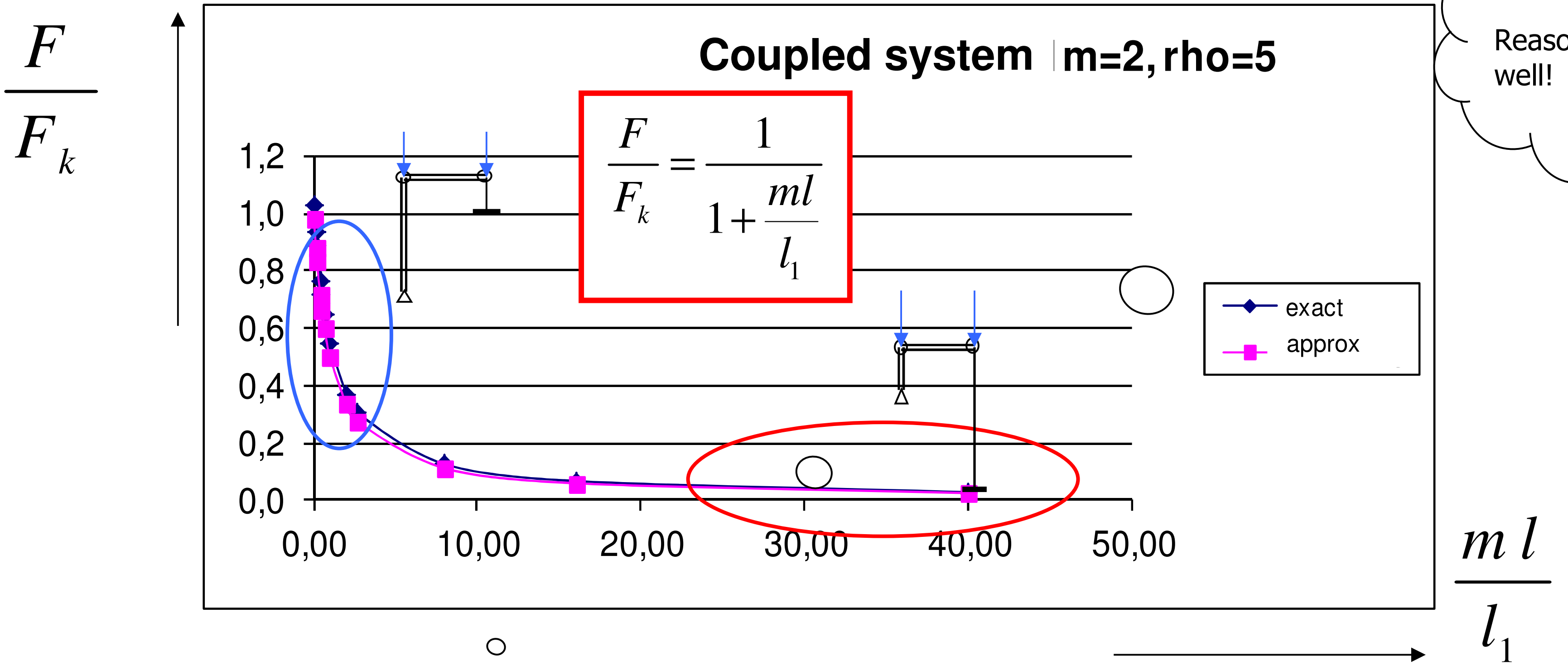
l_1/l	αl	β	γ
0,050	0,2136	14,7046	0,0277
0,125	0,3313	9,4814	0,0667
0,250	0,4544	6,9130	0,1255
0,750	0,7074	4,4410	0,3042
1,000	0,7801	4,0269	0,3699
2,000	0,9489	3,3105	0,5474
3,000	1,0352	3,0347	0,6515
4,000	1,088165	2,887055	0,7198
5,000	1,124096	2,794773	0,7681
10,000	1,208024	2,600605	0,8871
15,000	1,240433	2,532658	0,9354
100,00	1,302002	2,412894	1,0305

example:

$l_1 = 0,125l \quad \Rightarrow \quad \gamma = 0,0667$

short two-force element in **compression** reduces the buckling load!

COMPARISON WITH DESIGN APPROACH (*m* variabel)



CONCLUSION:

Reasonably well!

CONCLUSION

SIMPLE DESIGN FORMULA:

$$\frac{F}{F_k} = \frac{1}{1+\xi} \quad \text{with:} \quad \xi = \frac{m_1 l}{l_1} + \frac{m_2 l}{l_2} + \ldots$$

Remark: CONSERVATIVE result

Deviations to “exact results” are within a range of 5 – 12%

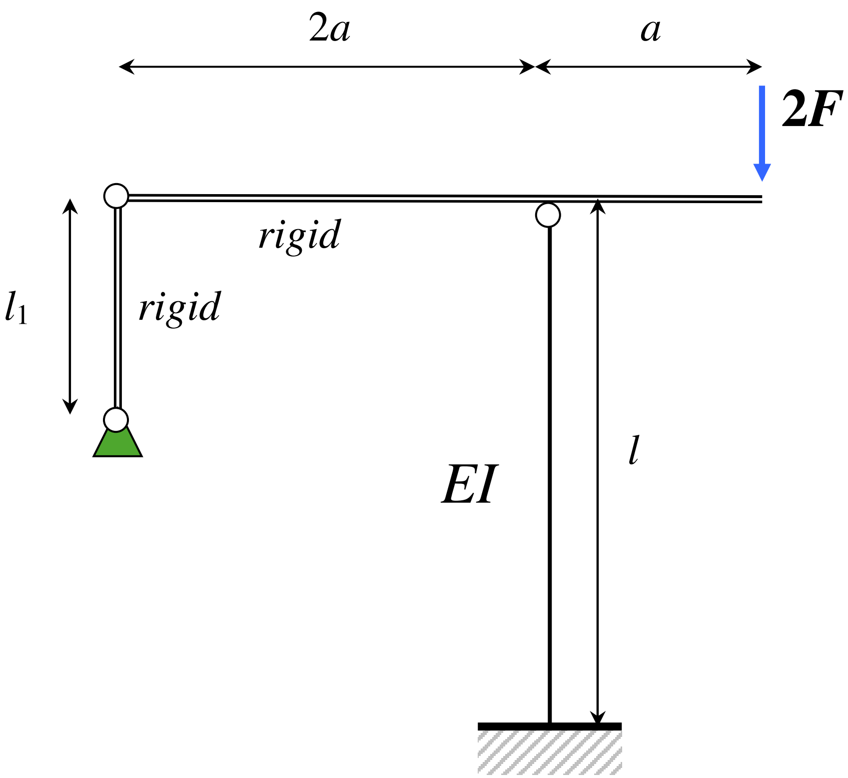
Remark

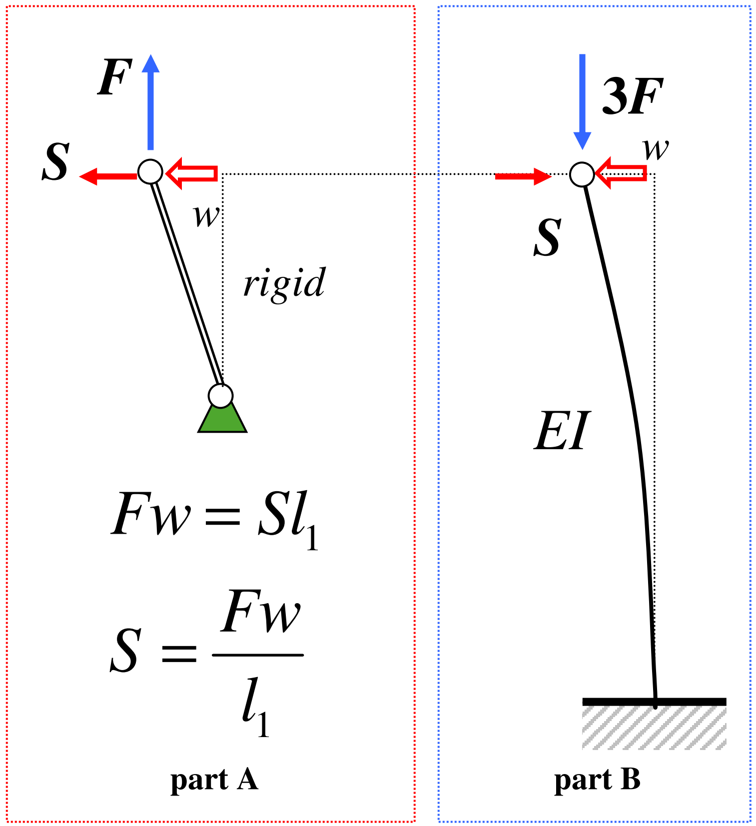
1 Exact results can be found with MAPLE/Python

2 In the book you will find even better approximations which can be used.

3 In this lecture it is not about the formula itself but about finding a useful formula.

FUN: CHECK ANOTHER COUPLED SYSTEM



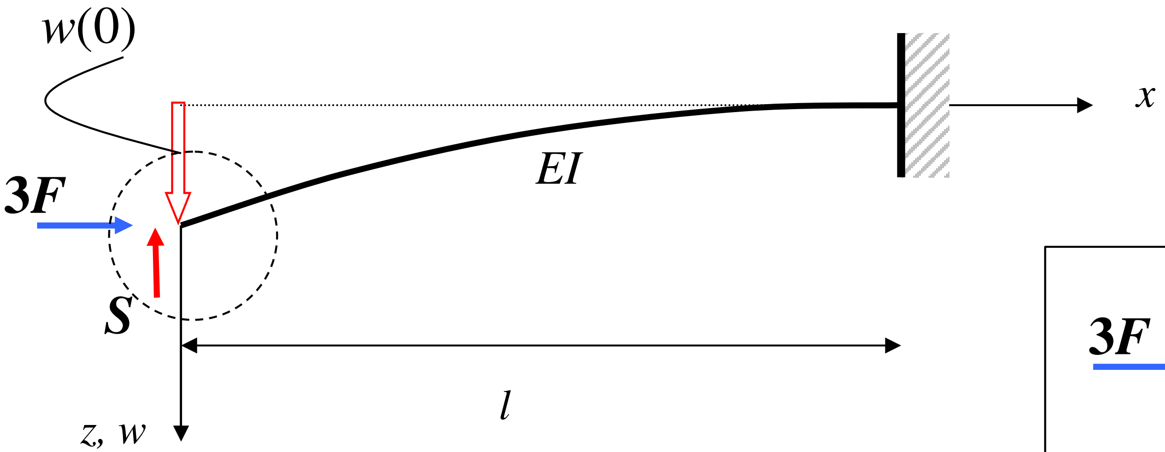


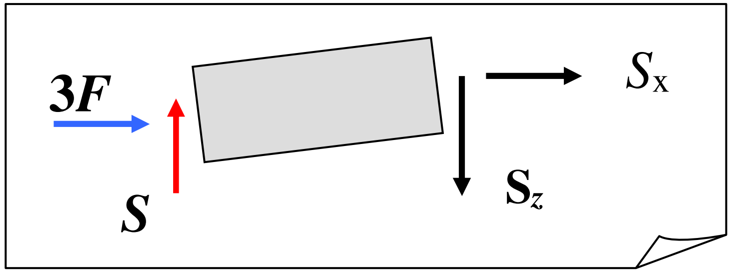
$$Fw = Sl_1$$

$$S = \frac{Fw}{l_1}$$

Repeat all previous steps to find the result …

EQUILIBRIUM OF PART B





$$w'''' + \alpha^2 w'' = 0 \quad \text{with:} \quad \alpha^2 = \frac{3F}{EI}$$

$$S_z = M' + S_x w' \Leftrightarrow S_z = -EIw''' - 3Fw'$$

Boundary conditions:

1) $S_z(0) = S$ 3) $w(l) = 0$

2) $M(0) = 0$ 4) $\varphi(l) = 0$

SOLVE

slope of the force line

Tooling:

$$w(x) = C_1 + C_2 x + C_3 \cos \alpha x + C_4 \sin \alpha x$$

$$w'(x) = C_2 - \alpha C_3 \sin \alpha x + \alpha C_4 \cos \alpha x \qquad S_z = -EIw''' - 3Fw' = -3FC_2$$

$$w''(x) = -\alpha^2 C_3 \cos \alpha x - \alpha^2 C_4 \sin \alpha x$$

Boundary conditions:

1) $-3FC_2 = \dfrac{F}{l_1}(C_1 + C_3)$

Use: $S = \dfrac{Fw(0)}{l_1}$

2) $EI\alpha^2 C_3 = 0 \quad \Rightarrow \quad C_3 = 0$

3) $C_1 + C_2 l + C_3 \cos \alpha l + C_4 \sin \alpha l = 0$

4) $C_2 - \alpha C_3 \sin \alpha l + \alpha C_4 \cos \alpha l = 0$

RESULT

$$\begin{bmatrix} F & 3Fl_1 & 0 \\ 1 & l & \sin \alpha l \\ 0 & 1 & \alpha \cos \alpha l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a homogeneous system of equations which has only a non-trivial solution when its determinant is equal to zero:

$$\tan \alpha l = (l - 3l_1)\alpha \quad \wedge \quad \alpha^2 = \frac{3F}{EI}$$

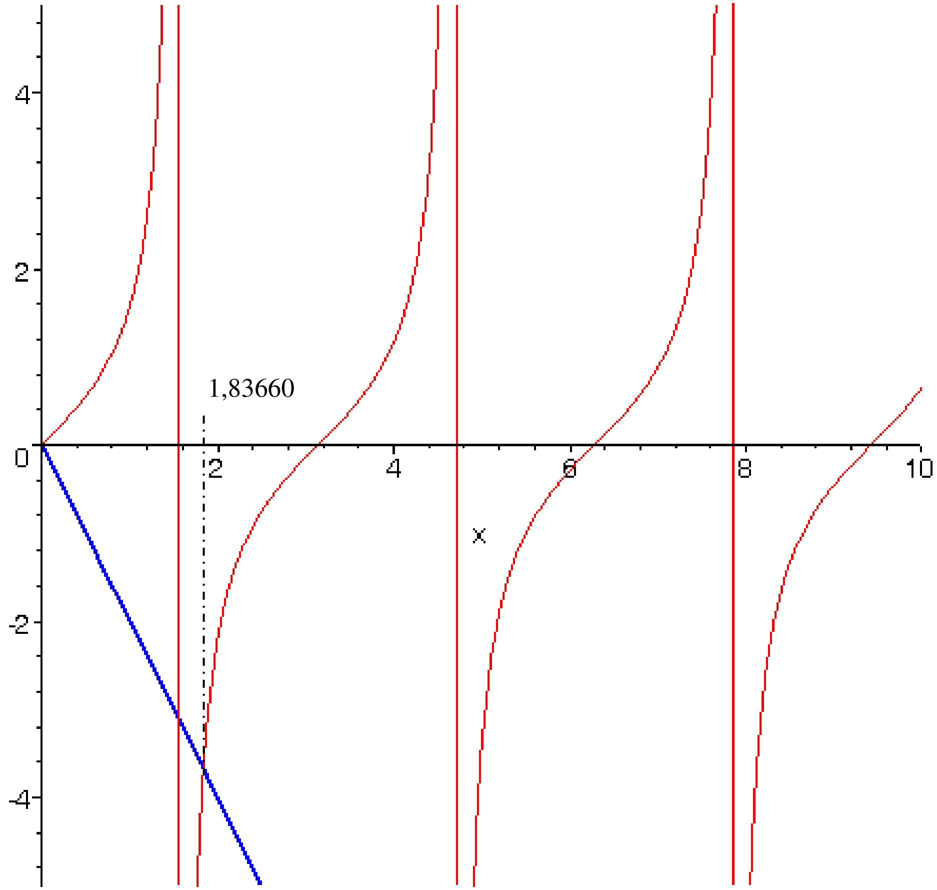
TRANSCENDENTAL EQUATION

SOLUTION for $l_1 = l$

$$\alpha l \cong 1{,}836601 \cong \frac{11}{6}$$

$$3F_k = \frac{EI}{\left(\frac{6}{11}\right)^2 l^2} = \frac{\pi^2 EI}{\left(\frac{6\pi}{11} l\right)^2}$$

$$l_k = \frac{6\pi}{11} l \cong 1{,}7l$$



Buckling length reduces to less than $2l$

short two-force element in **tension** increases the buckling load!