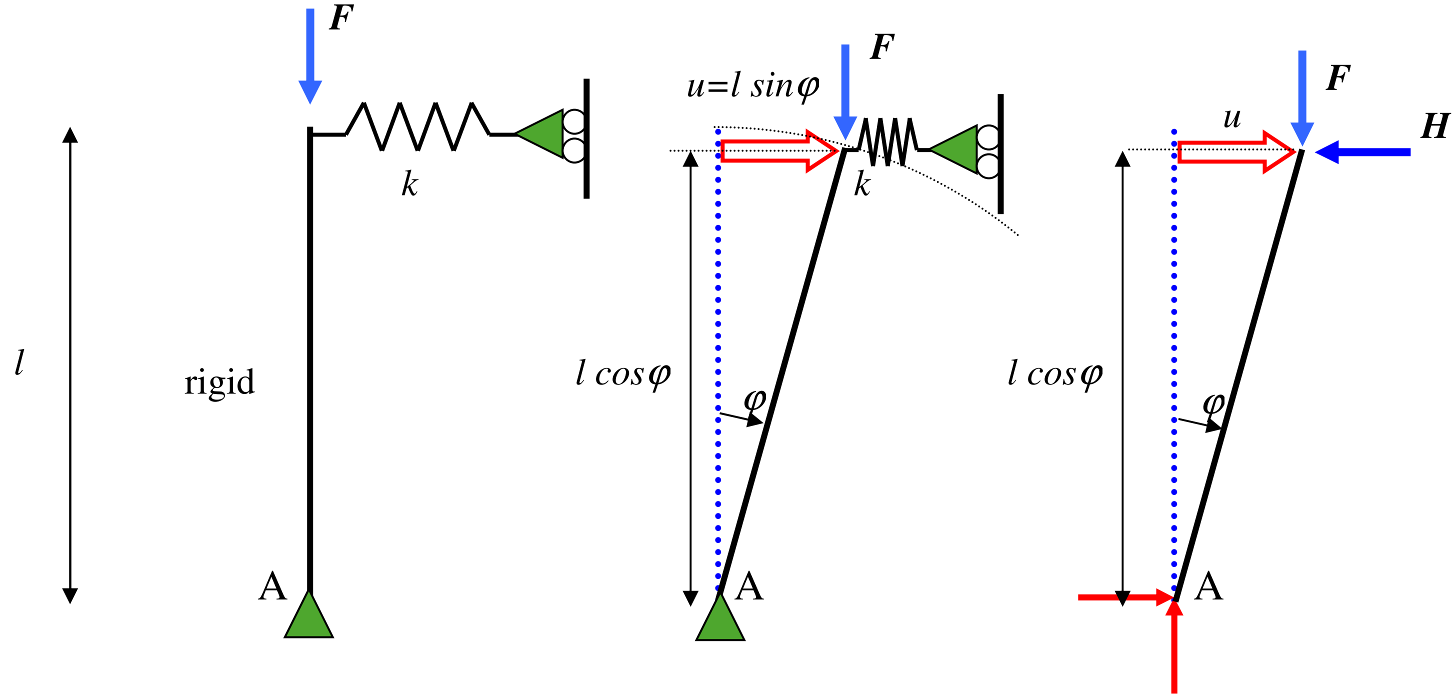
OBJECTIVES OF THIS PART

- 6 2^{nd} order effect and the magnification factor
  - Post-buckling
  - Initial displacement and second order effects – rigid models
  - Initial displacement and second order effects – flexural models

- Stability of equilibrium for large displacements for rigid models
- Examine increase of initial displacement due to action of compressive force in displaced situation for rigid rods
- Model this effect with magnification factor
- Apply this on a structural model
- Check the model for flexural buckling

POST BUCKLING BEHAVIOUR, rigid model 1

(large displacements)



Structure with loading *Deformed situation* *Free body*

EQUILIBRIUM (large displacements)

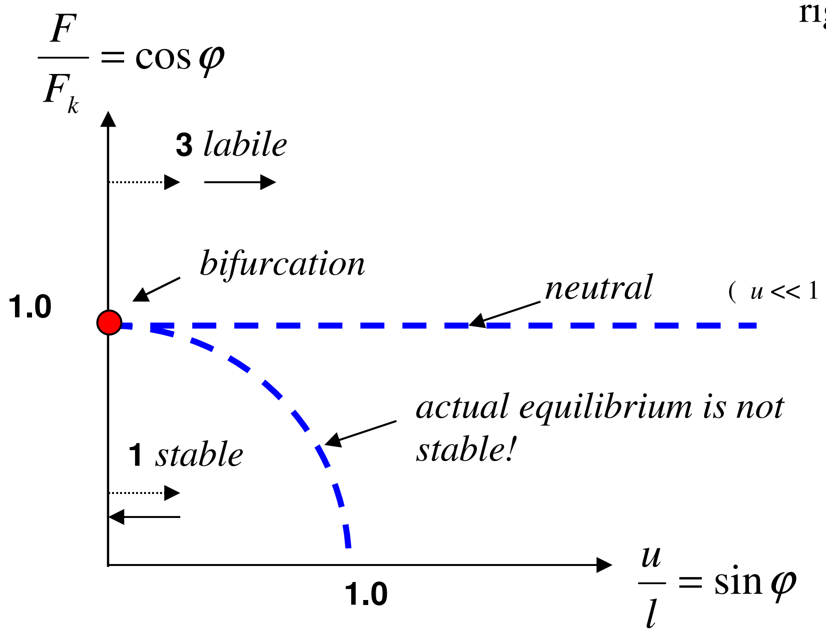
$$F\,u = Hl\cos\varphi$$

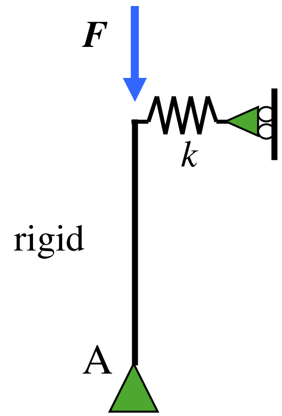
$$F\,u = k\,u\,l\cos\varphi$$

$$F = k\,l\cos\varphi$$

with: $F_k = k\,l$

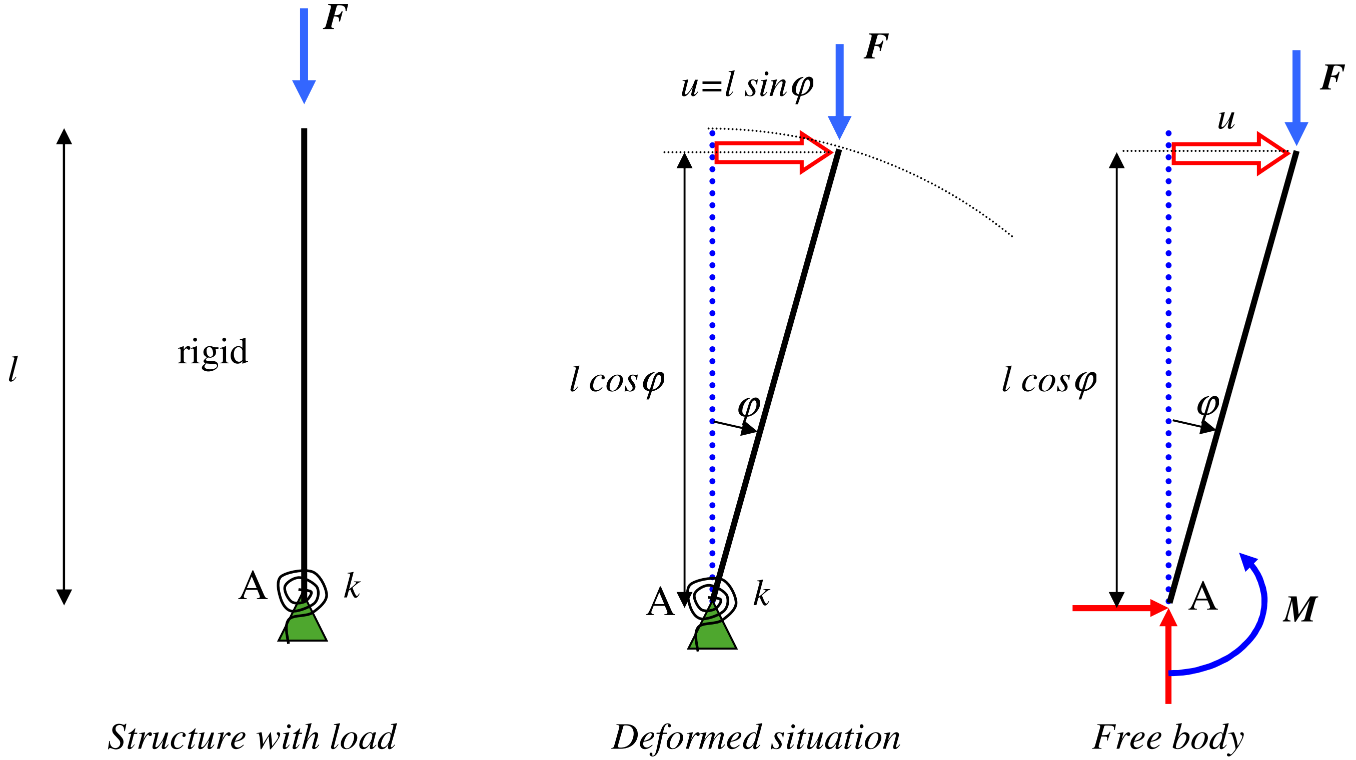
$$\frac{F}{F_k} = \cos\varphi$$





POST BUCKLING BEHAVIOUR, rigid model 2

(large displacements)



EQUILIBRIUM (large displacements)

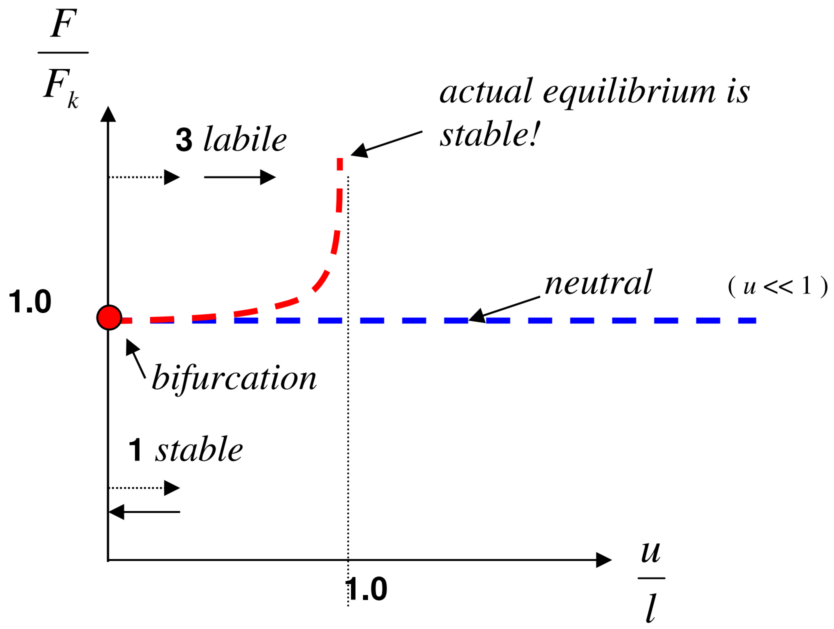
$$F\,u = M$$

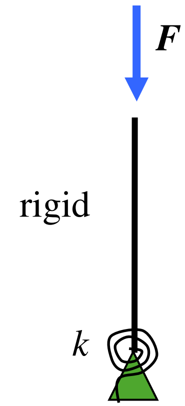
$$F\,l\sin\varphi = k\,\varphi$$

$$F = \frac{k}{l}\frac{\varphi}{\sin\varphi}$$

with: $F_k = \dfrac{k}{l}$

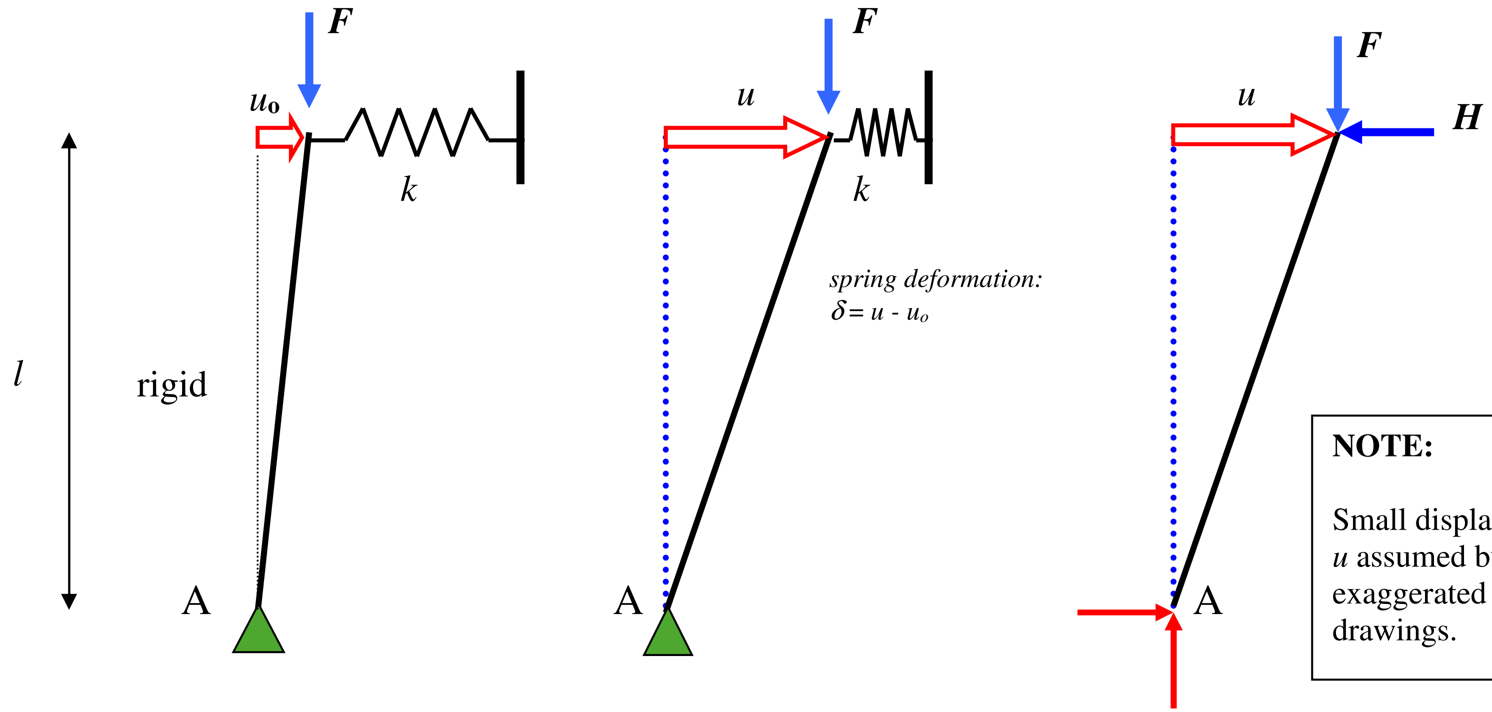
$$\frac{F}{F_k} = \frac{\varphi}{\sin\varphi}$$





INITIAL DISPLACEMENT (magnification factor)

(Chapter 10) (small displacements, rigid model)



Structure with load *Deformed situation* *Free body*

NOTE:

Small displacements *u* assumed but exaggerated in the drawings.

ELABORATE

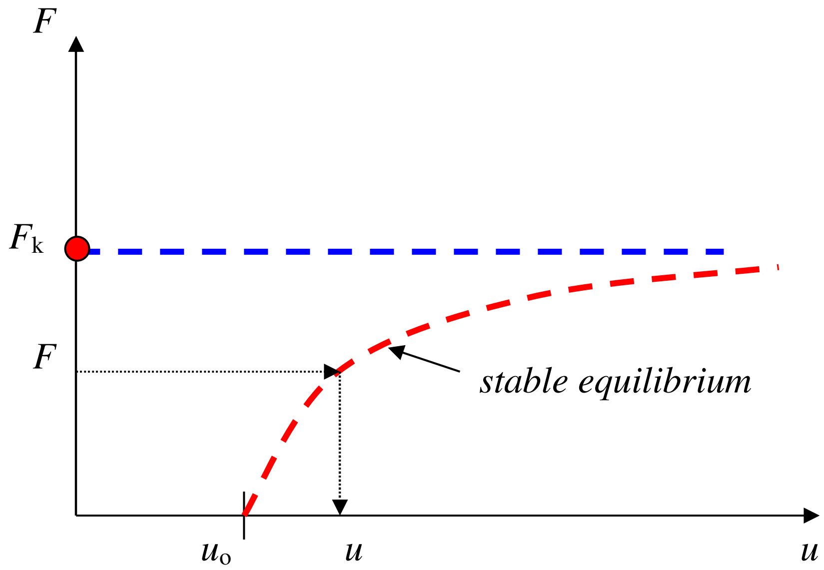
$$F\,u = H\,l = k\,l\,(u - u_o) \quad \Leftrightarrow$$

$$\frac{F}{F_k} = \frac{u - u_o}{u} = 1 - \frac{u_o}{u} \quad \Leftrightarrow$$

$$u = \left(\frac{F_k}{F_k - F} \right) u_o \qquad define: n = \frac{F_k}{F}$$

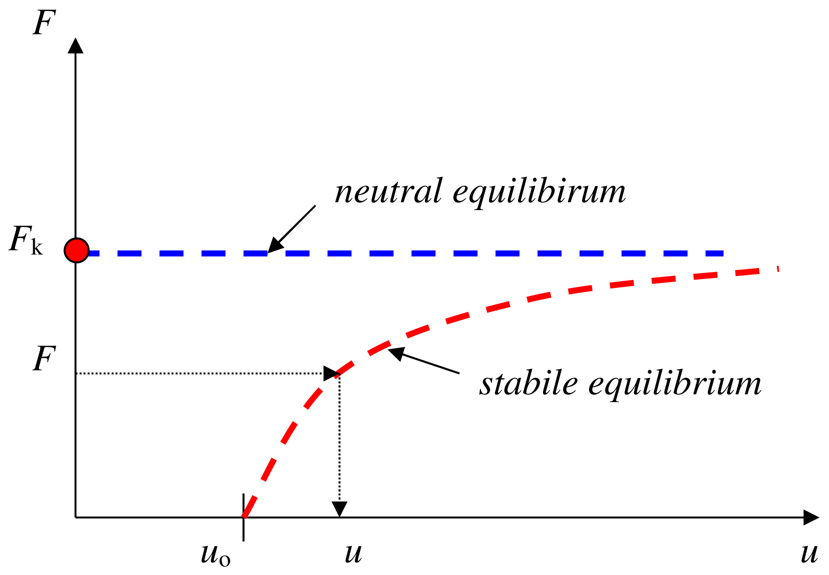
$$u = \left(\frac{n}{n-1} \right) u_o$$

magnification factor



IMPORTANT CONCEPTS

(found for rigid models)



- Buckling load
- Initial displacement
  - 1st order displacement, linear calculation
  - 2nd order dislacement, Geometrical non-linear calculation (GNL)
  - Magnification factor

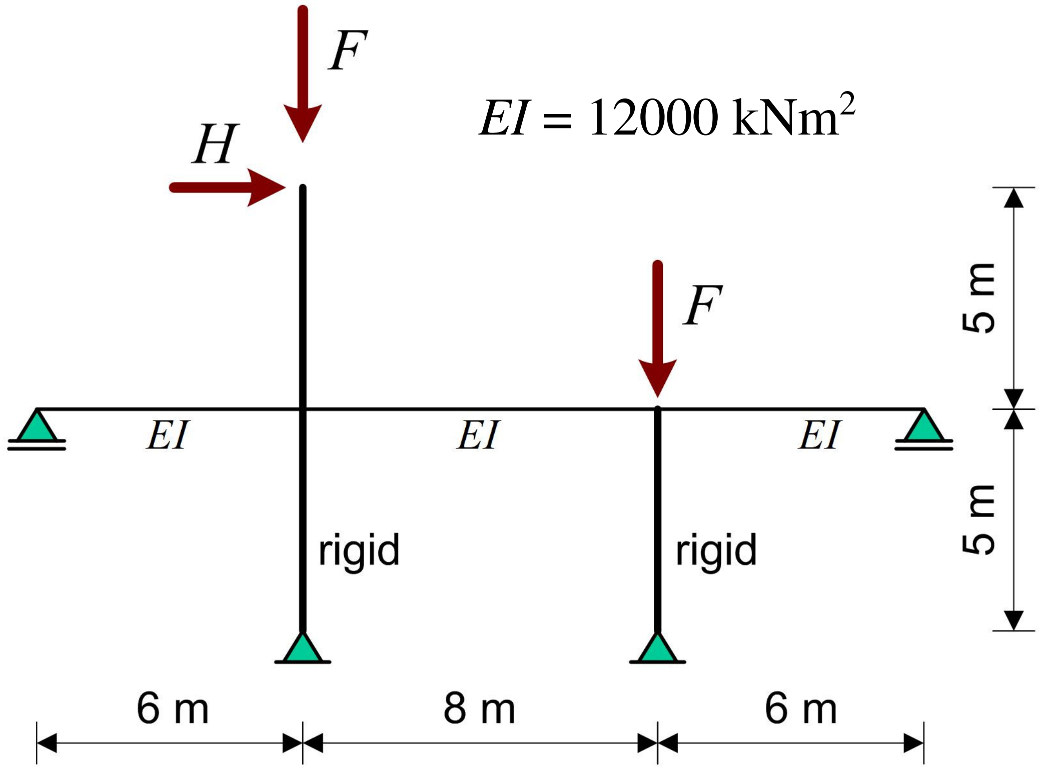
EXAMPLE: INITIAL DISPLACEMENT due to a horizontal load

assume :

$u \ll 1$

Question:

- Critical load F_c
- Find u for for $H = 12$ kN; $F = 0$ kN (1st order calculation)
- Find u for $H = 12$ kN ; $F = 1000$ kN (2nd order calculation)
- Draw *load - displacement* diagram (for $H = 12$ kN)

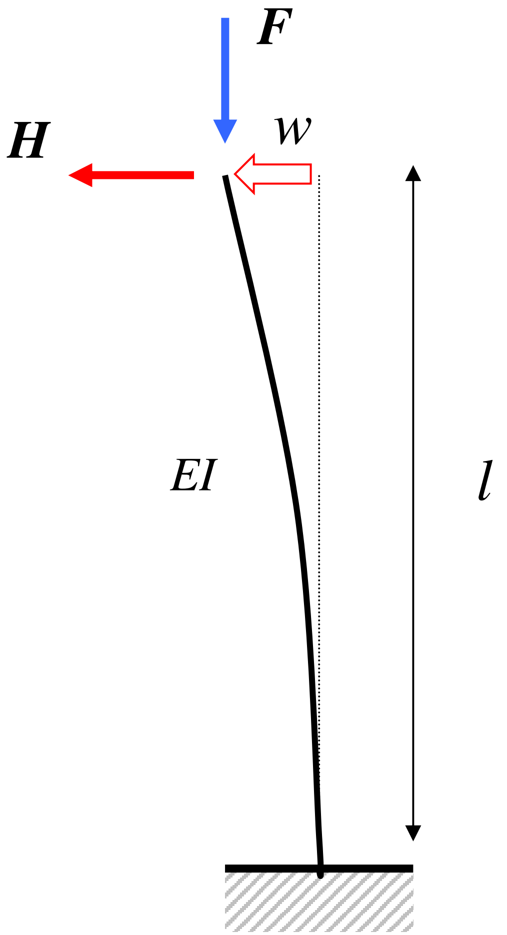


MAGNIFICATION FACTOR, flexural models

(Chapter 11)

EXAMPLE

Find the influence of the vertical load *F* on the horizontal deflection *w*, initiated by *H*.



1st order deflection:

$$w_o = \frac{Hl^3}{3EI} \quad \Rightarrow \quad H = \frac{3EI}{l^3} w_o$$

TOOLING 2nd order calculation

$$EIw'''' + Fw'' = 0 \quad \left(q_x, q_z = 0\right)$$

and

$$S_z = M' - Fw'$$

General solution:

$$w(x) = \mathrm{C}_1 + C_2 x + C_3 \cos\alpha x + C_4 \sin\alpha x$$

- 4th order ODE
- 4 boundary conditions

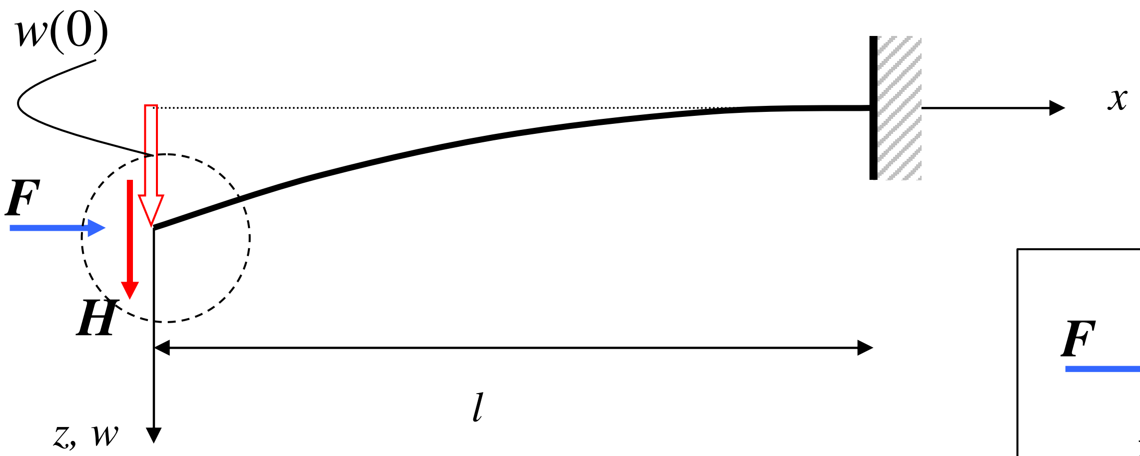
Some derivatives to elaborate boundary conditions:

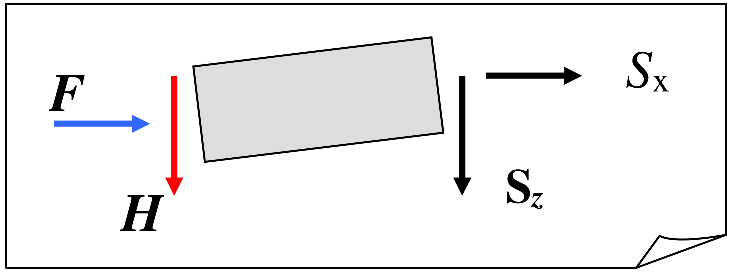
$$w(x) = C_1 + C_2 x + C_3 \cos\alpha x + C_4 \sin\alpha x$$

$$w'(x) = C_2 - \alpha C_3 \sin\alpha x + \alpha C_4 \cos\alpha x \qquad S_z = -EIw''' - Fw' = -FC_2$$

$$w''(x) = -\alpha^2 C_3 \cos\alpha x - \alpha^2 C_4 \sin\alpha x$$

EQUILIBRIUM





$$w'''' + \alpha^2 w'' = 0 \quad \text{with:} \quad \alpha^2 = \frac{F}{EI}$$

$$S_z = M' - Fw' \Leftrightarrow S_z = -EIw''' - Fw'$$

Boundary conditions:

1) $S_z(0) = -H$ 3) $w(l) = 0$

2) $M(0) = 0$ 4) $\varphi(l) = 0$

System of equations from BC:

1) $-FC_2 = -H \quad \Rightarrow \quad C_2 = H/F$

2) $EI\alpha^2 C_3 = 0 \quad \Rightarrow \quad C_3 = 0$

3) $C_1 + C_2 l + C_3 \cos\alpha l + C_4 \sin\alpha l = 0$

4) $C_2 - \alpha C_3 \sin\alpha l + \alpha C_4 \cos\alpha l = 0$

Result:

1) $C_2 = H/F$

2) $C_3 = 0$

3) $C_1 = \frac{H}{F}\left(\frac{\tan\alpha l}{\alpha} - l\right)$

4) $C_4 = -\frac{H}{F\alpha\cos\alpha l}$

SOLUTION:

$$w(x) = \frac{H}{F}\left(\frac{\tan\alpha l}{\alpha} - l\right) + \frac{H}{F}x - \frac{H}{F}\frac{\sin\alpha x}{\alpha\cos\alpha l} \qquad \text{with: } \alpha^2 = \frac{F}{EI}$$

EXACT DEFLECTION IS FOUND FOR GIVEN *F* and *H* !

Not a buckling problem but a 2nd order calculation!

2nd ORDER DEFLECTION AT FREE END $(x = 0)$:

$$w(0) = \frac{H}{F}\left(\frac{\tan\alpha l}{\alpha} - l\right) \quad \text{with:} \quad H = \frac{3EI}{l^3} w_o$$

$$w(0) = \frac{Hl}{F}\left(\frac{\tan\alpha l}{\alpha l} - 1\right) = \frac{3EI}{l^2 F}\left(\frac{\tan\alpha l}{\alpha l} - 1\right) w_o$$

$$w(0) = \boxed{\frac{3}{(\alpha l)^2}\left(\frac{\tan\alpha l}{\alpha l} - 1\right)} w_o$$

exact magnification factor

MAGNIFICATION FACTOR

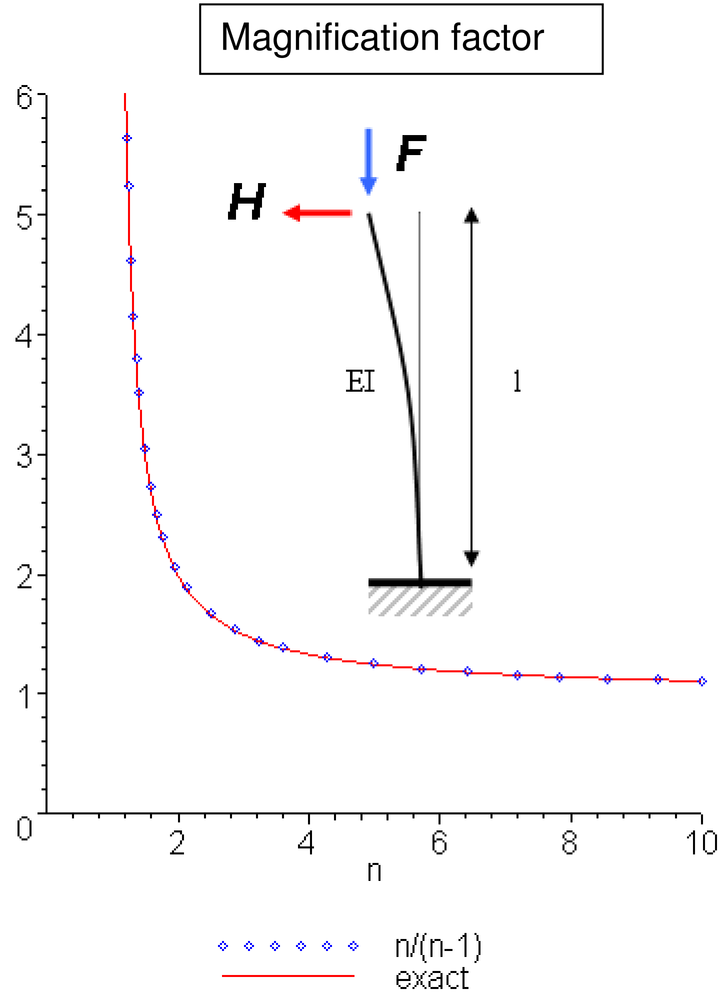
$$\text{rigid models} = \frac{n}{n-1}$$

$$\text{exact} = \frac{3}{(\alpha l)^2}\left[\frac{\tan(\alpha l)}{\alpha l} - 1\right]$$

with:

$$n = \frac{F_k}{F} = \frac{\pi^2 EI}{4l^2\alpha^2 EI} = \frac{\pi^2}{4(\alpha l)^2}$$

$$(\alpha l) = \frac{\pi}{2\sqrt{n}}$$



Conclusion: $\frac{n}{n-1}$ fits very well?

RESULTS EXAMPLE

n	αl	$exact$	$n/(n\text{-}1)$
1,01	1,56	99,56	101,00
1,05	1,53	20,71	21,00
1,25	1,40	4,94	5,00
1,50	1,28	2,97	3,00
1,75	1,19	2,31	2,33
2,00	1,11	1,99	2,00
2,50	0,99	1,66	1,67
5,00	0,70	1,25	1,25
7,50	0,57	1,15	1,15
10,00	0,50	1,11	1,11

Good match but with small differences.

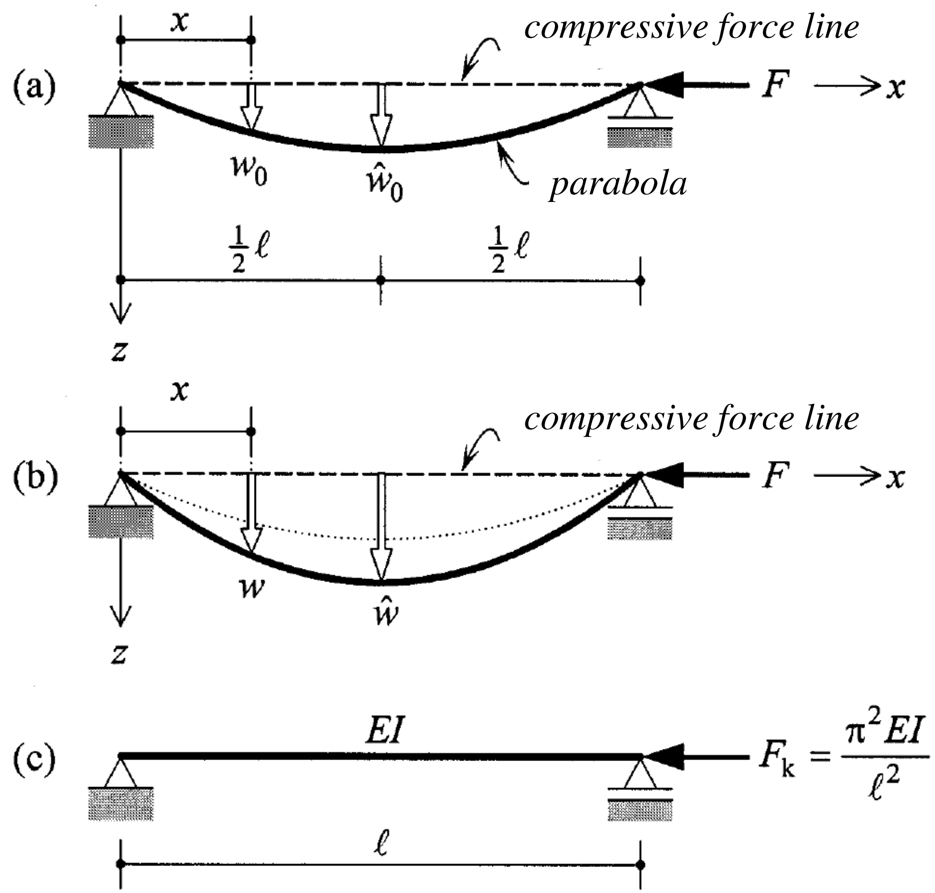
Is there an explanation?

EXAMPLE (par 11.2.2)

- parabolic initial deflection

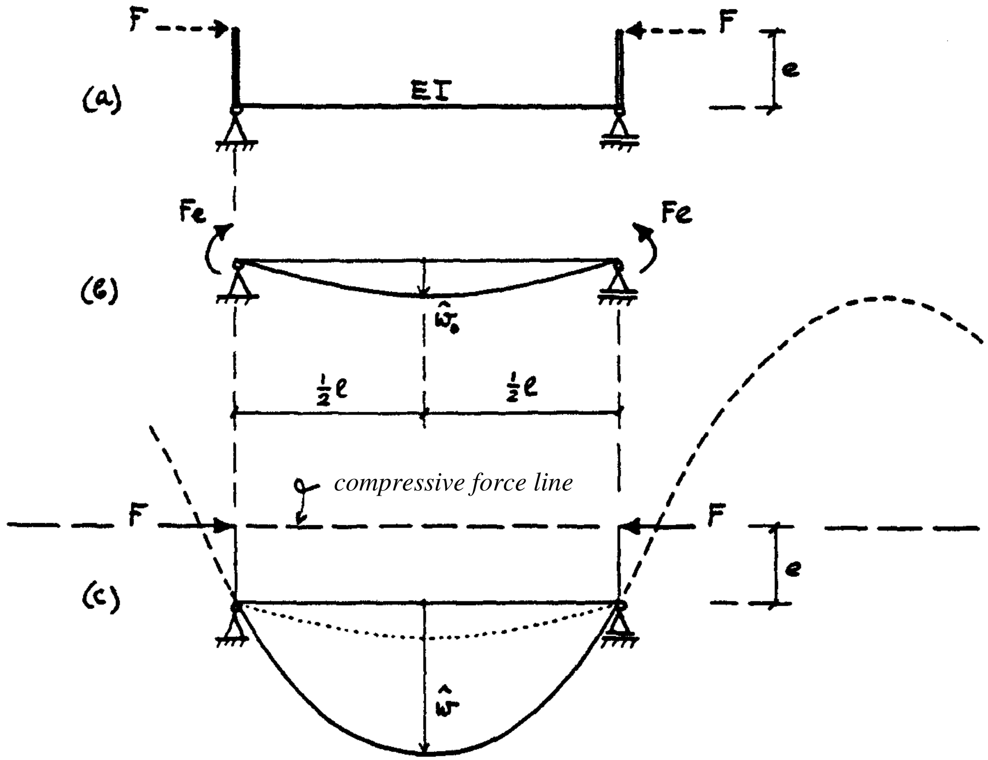
Very good approximation:

$$\frac{n}{n-1}$$



Conclusion: $\frac{n}{n-1}$ fits very well?

EXAMPLE
Excentric compressive force

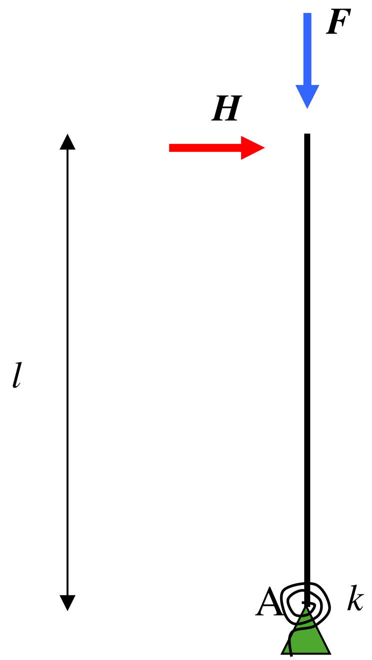


Approximation:

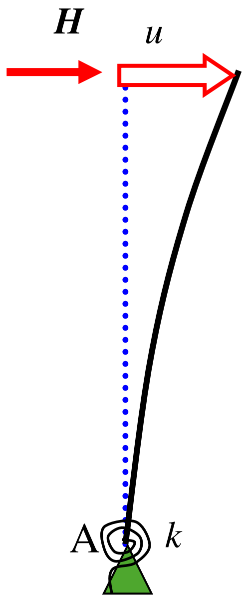
Magnification factor for *bending moment* at midspan:

$$\frac{n+0{,}25}{n-1}$$

EXAMPLE 2nd order BENDING MOMENT (par 11.3.3)



Structure with loads



Linear result, 1st order

1st order deflection:
(due to H:)

$$u_o = \frac{Hl^3}{3EI} + \frac{(Hl)\,l}{r} \quad define: \rho = \frac{kl}{EI}$$

$$u_o = Hl \times \frac{l^2}{3EI}\left(1 + \frac{3}{\rho}\right)$$

$$u_o = \frac{l^2}{3EI}\left(1 + \frac{3}{\rho}\right) M_o \quad with: \quad M_o = Hl$$

2^{nd} ORDER MOMENT

Moment in the spring due to the total load in the deformed situation (GNL).

2^{nd} order moment in the rotational spring:

$$M = Hl + F\,u = Hl + F\frac{n}{n-1}u_o = Hl + \frac{nF}{n-1}Hl\frac{l^2}{3EI}\left(1+\frac{3}{\rho}\right)$$

$$M = M_o\left(1+\frac{nF}{n-1}\frac{l^2}{3EI}\left(1+\frac{3}{\rho}\right)\right) = M_o\left(\frac{n-1+nF\dfrac{l^2}{3EI}\left(1+\dfrac{3}{\rho}\right)}{n-1}\right) \quad \text{with:} \quad n=\frac{F_k}{F}$$

$$M = M_o\left(\frac{n-1+\dfrac{F_k l^2}{3EI}\left(1+\dfrac{3}{\rho}\right)}{n-1}\right) \qquad \text{with:} \quad F_k = \frac{\pi^2 EI}{l^2\left(4+\dfrac{\pi^2}{\rho}\right)}$$

ELABORATE THIS RESULT

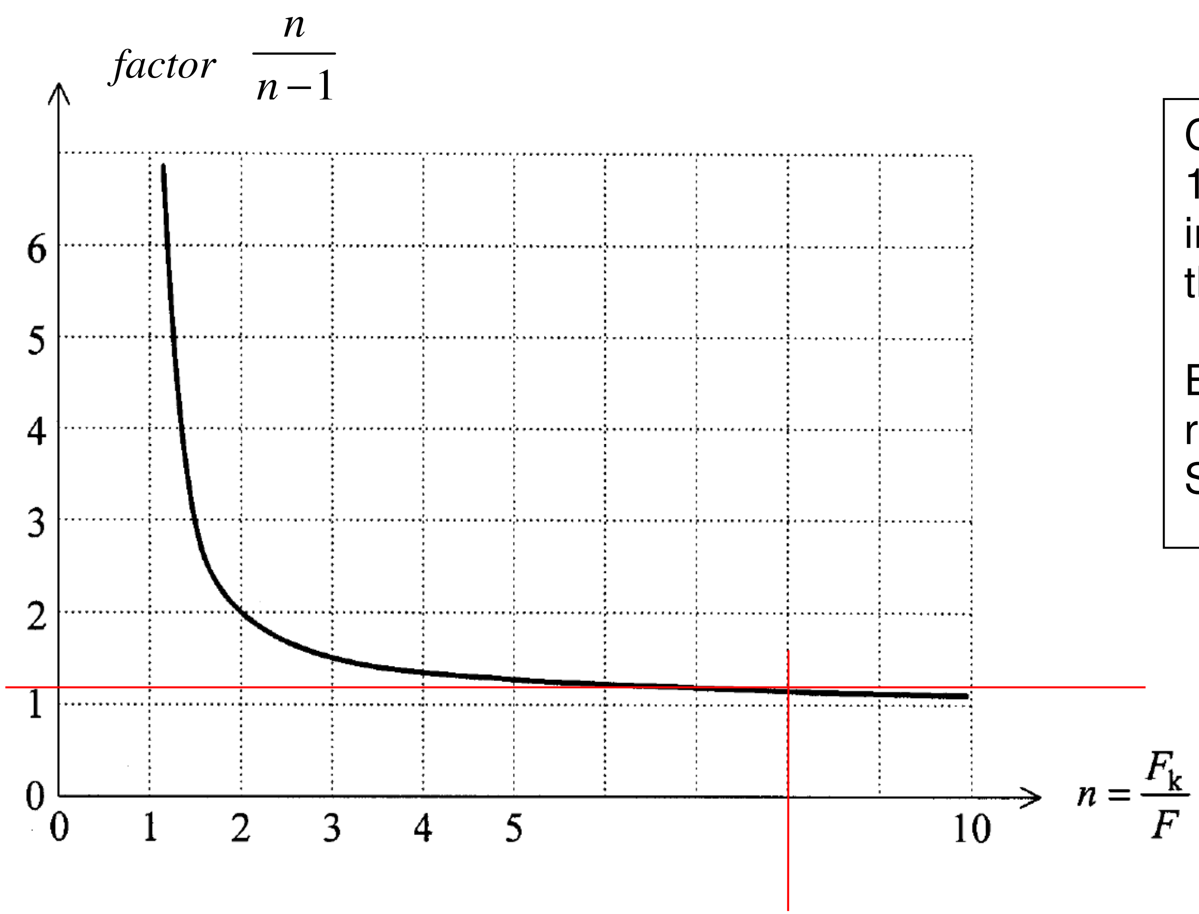
2nd order moment in the spring:

$$M = M_o \left(\frac{n - 1 + \dfrac{\pi^2 EIl^2 \left(1 + \dfrac{3}{\rho}\right)}{3EIl^2 \left(4 + \dfrac{\pi^2}{\rho}\right)}}{n-1} \right) = M_o \left(\frac{n - 1 + \dfrac{\pi^2 (1 + \frac{3}{\rho})}{3(4 + \frac{\pi^2}{\rho})}}{n-1} \right) \qquad \text{with: } \pi^2 \approx 10$$

Not expected ...

$$M = \frac{n - \dfrac{0,2\rho}{1.2\rho + 3}}{n-1} M_o \qquad \textit{magnification} \text{ factor for the moment : } \frac{n - \dfrac{\rho}{3(5 + 2\rho)}}{n-1}$$

ACCEPTABLE MAGNIFICATION FACTOR



Choose as a limit max 15% magnification of the initial displacement due to the compressive force.

Engineering code has rules for this, see lectures Steel and Concrete.

$$\frac{n}{n-1} \leq 1{,}15 \quad \Leftrightarrow n \geq 8 \quad \Rightarrow F_{max} \leq 0.125 F_k \quad (\max 12.5\% \; of \; F_k \; !!)$$

CONCLUSIONS

MAGNIFICATION FACTOR USING FLEXURAL MODELS

- Standard approach for rigid models is sufficient accurate,
- Although the factor $\frac{n}{n-1}$ is not exact since 2nd order defletion is not affine with 1st order deflection,
- Pay attention for some special cases in which end-moments depend on the compressive force,
- Limits to the influence of the magnification reduced allowed compressive force considerable.
  (never a design load near the buckling load)
  (a very exact calculation of the buckilng load becomes then also less relevant)

EXAMPLES