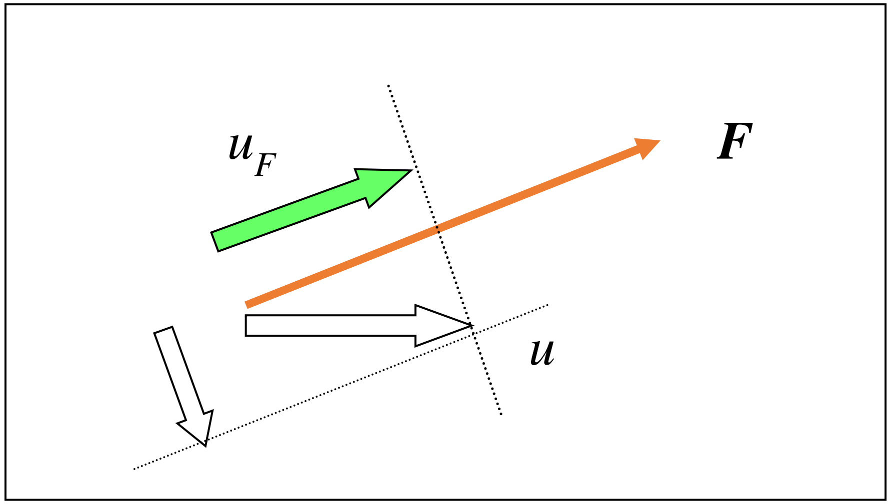
OBJECTIVES OF THIS PART

- 7 Rayleigh approximation method for flexural buckling
  - Work and Energy
  - Clapeyron's principle
  - Rayleigh's approach
  - Examples

WORK

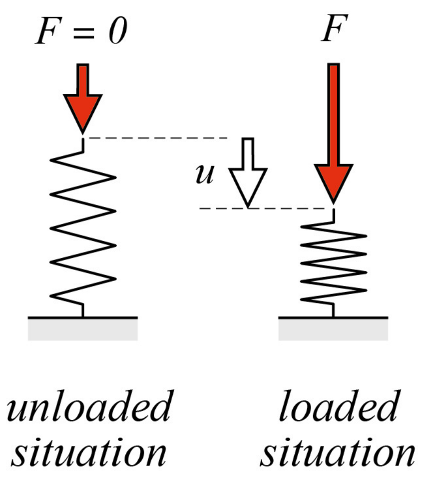


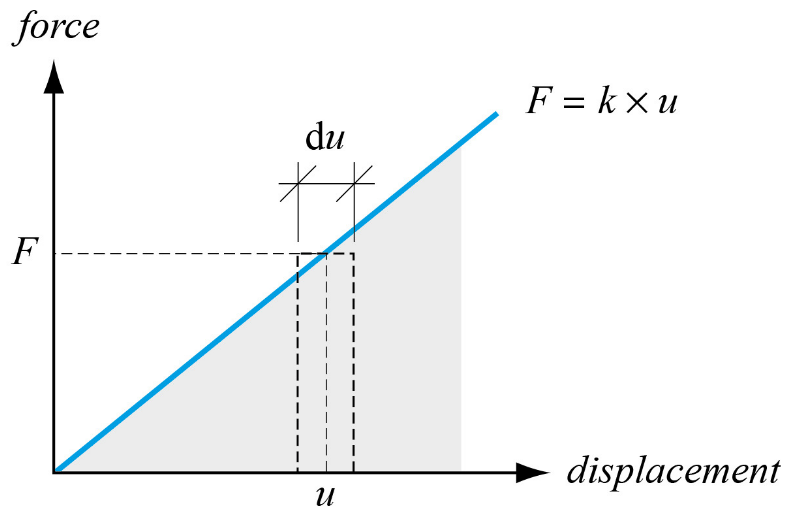
$$A = \vec{F} \cdot \vec{u} \quad \text{or} \quad A = F u_F$$

scalar

Unit of work is *Joule* $\left[\frac{kg \cdot m}{s^2} \cdot m = N \cdot m = W \cdot s \right]$

STRAIN ENERGY (POTENTIAL ENERGY)



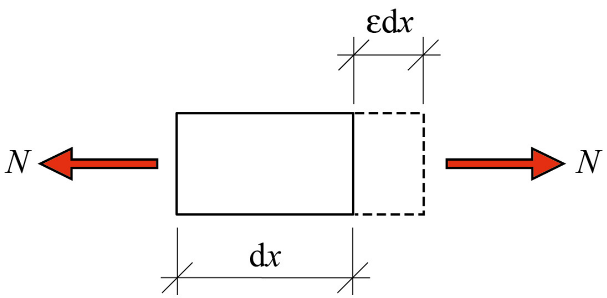


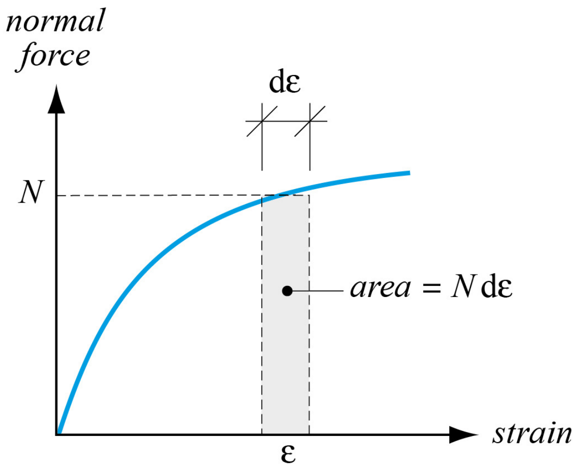
spring behaviour

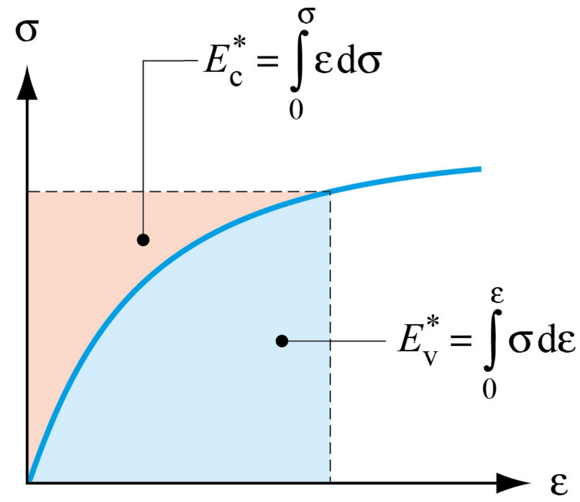
$$E = \frac{1}{2} k u^2 \qquad \left[\frac{kg \cdot m^2}{s^2} = N \cdot m \right]$$

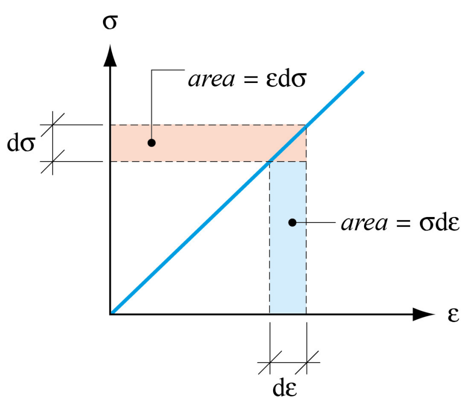
STRAIN ENERGY for a ROD

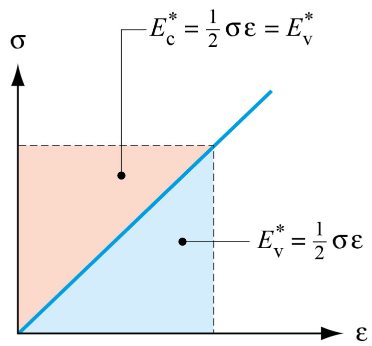
per unit length for basic cases











$$E_v^* = \tfrac{1}{2} EA\varepsilon^2$$

LE

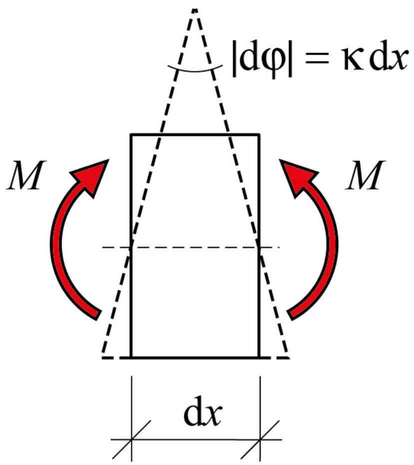
or

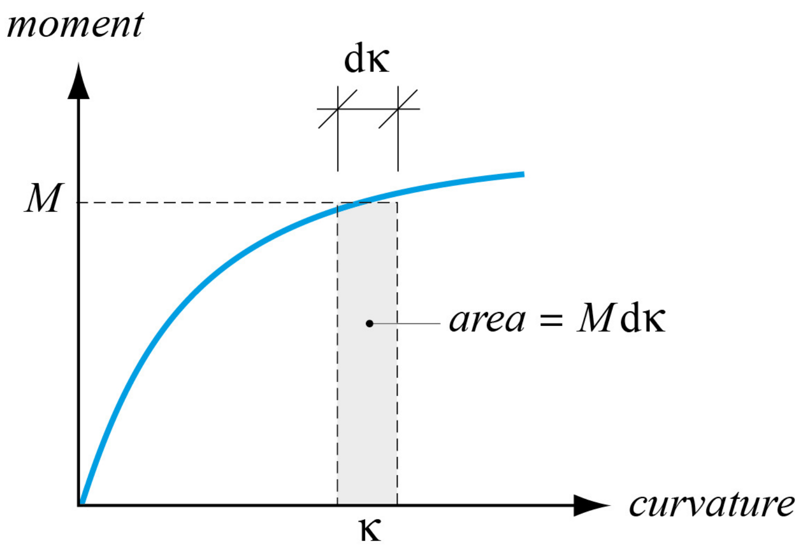
$$E_c^* = \frac{N^2}{2EA}$$

$$\left[\frac{\frac{kg \cdot m}{s^2}}{m}\right] = [N]$$

STRAIN ENERGY for an EULER BEAM

per unit length for basic cases



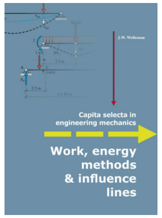


LE

$$E_v^* = \tfrac{1}{2} EI \kappa^2 \quad or$$

$$E_c^* = \frac{M^2}{2EI} \quad \left[\frac{\frac{kg \cdot m}{s^2}}{m} \right] = [N]$$

SUMMARY (see book[1])

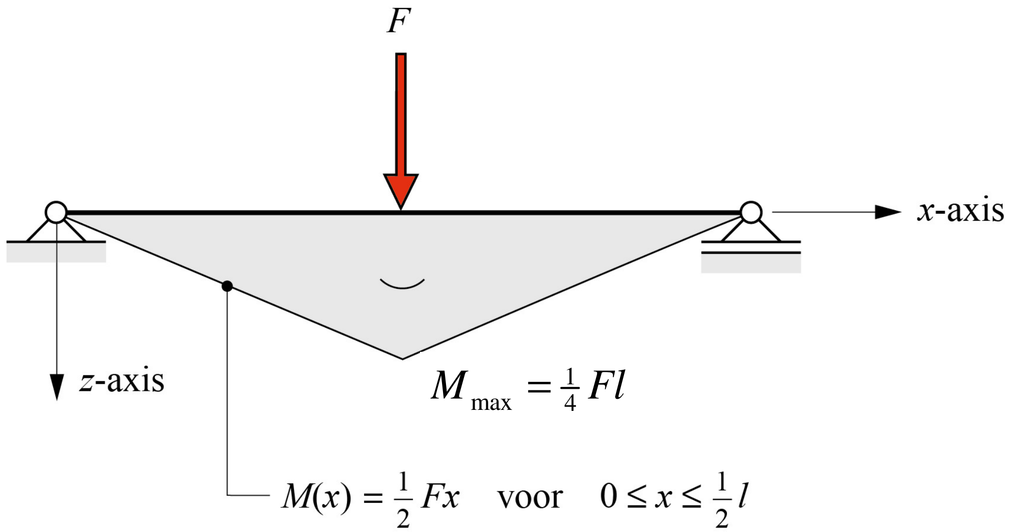


	Stress quantity (complementary)	*Strain quantity*	*combination*
Extension	$E_c^* = \dfrac{N^2}{2EA}$	$E_v^* = \tfrac{1}{2} EA\varepsilon^2$	$E_v^* = \tfrac{1}{2} N\varepsilon$
Shear	$E_c^* = \dfrac{V^2}{2GA_s}$	$E_v^* = \tfrac{1}{2} GA_s\gamma^2$	$E_v^* = \tfrac{1}{2} V\gamma$
Bending	$E_c^* = \dfrac{M^2}{2EI}$	$E_v^* = \tfrac{1}{2} EI\kappa^2$	$E_v^* = \tfrac{1}{2} M\kappa$
Torsion	$E_c^* = \dfrac{M_w^2}{2GI_w}$	$E_v^* = \tfrac{1}{2} GI_w\theta^2$	$E_v^* = \tfrac{1}{2} M_w\theta$
Normal Stress	$E_c^* = \dfrac{\sigma^2}{2E}$	$E_v^* = \tfrac{1}{2} E\varepsilon^2$	$E_v^* = \tfrac{1}{2} \sigma\varepsilon$
Shear Stress	$E_c^* = \dfrac{\tau^2}{2G}$	$E_v^* = \tfrac{1}{2} G\gamma^2$	$E_v^* = \tfrac{1}{2} \tau\gamma$

1 J.W. Welleman, *Work, energy methods & influence lines* (Capita selecta in engineering mechanics), Bouwen met Staal, Zoetermeer 2016, ISBN 978-90-72830-95-1,

EXAMPLE STRAIN ENERGY for an EULER BEAM

Simple supported beam with concentrated load



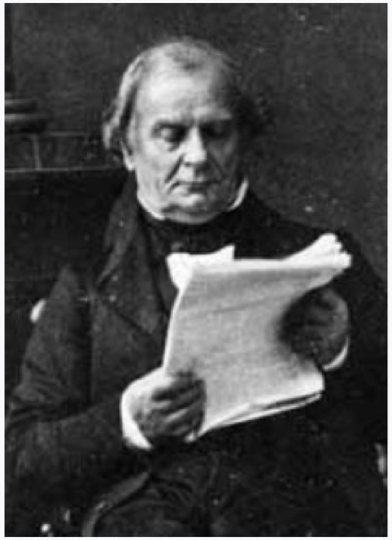
$$E = \int_{x=0}^{x=l} E_c^* \mathrm{d}x = \int_{x=0}^{x=l} \frac{M^2(x)}{2EI} \mathrm{d}x$$

$$E = 2 \int_{x=0}^{x=\frac{1}{2}l} \frac{\left(\frac{1}{2} Fx\right)^2}{2EI} \mathrm{d}x = \frac{\frac{1}{4} F^2}{EI} \left[\frac{1}{3} x^3 \right]_0^{\frac{1}{2}l} = \frac{F^2 l^3}{96EI} \quad [\mathrm{Nm}]$$

PRINCIPLE OF CLAPEYRON

Work done by loads is stored as strain energy (conservation of energy)

Émile Clapeyron

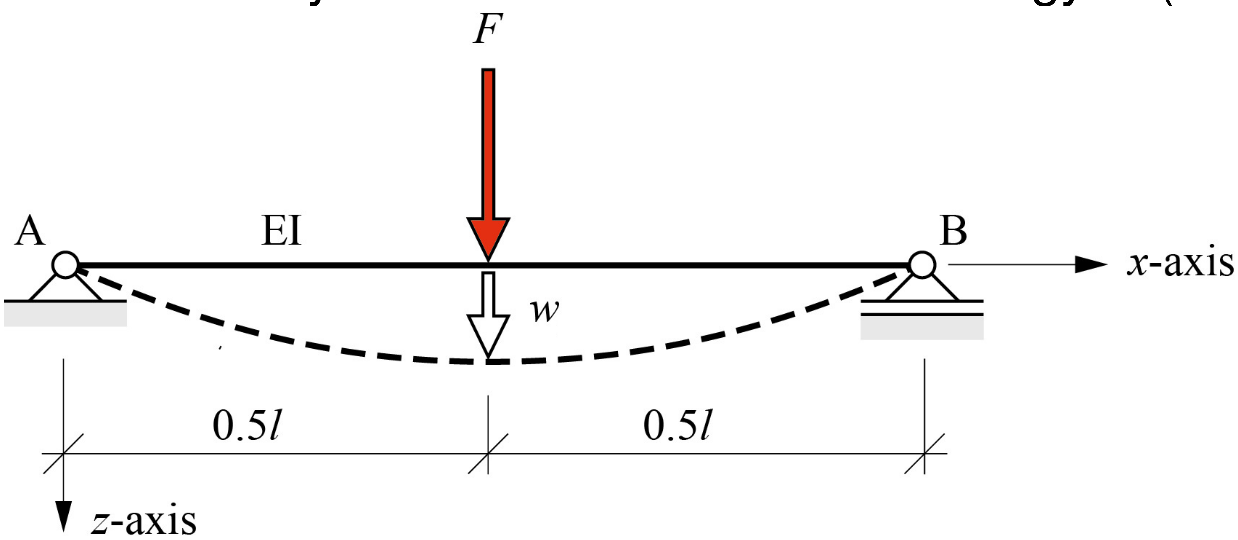


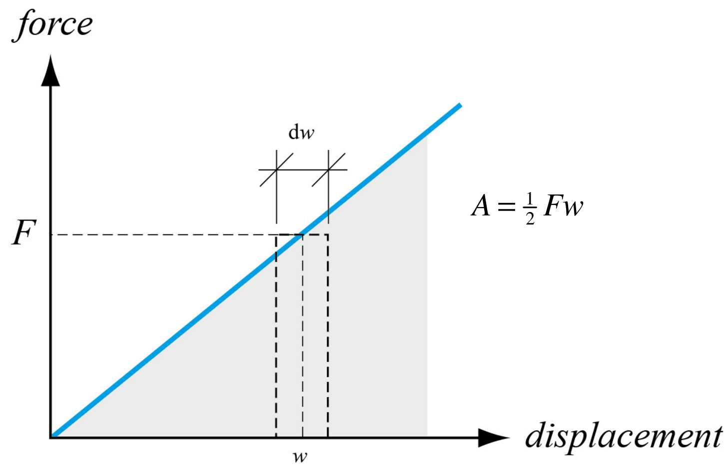
Born	26 January 1799 Paris, France
Died	28 January 1864 (aged 65) Paris, France
Nationality	French

Source : Wikipedia

PRINCIPLE OF CLAPEYRON

Work done by loads is stored as strain energy (conservation of energy)





$$E = \frac{F^2 l^3}{96EI} \quad [\text{Nm}]$$

$$A = \tfrac{1}{2} F w \quad [\text{Nm}]$$

Clapeyron :

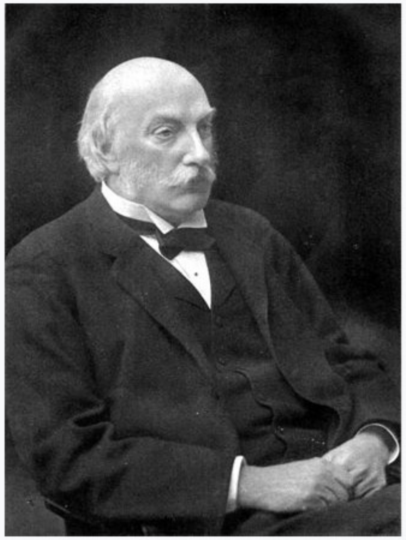
$$E = A \quad \Leftrightarrow$$

$$\frac{F^2 l^3}{96EI} = \tfrac{1}{2} F w \quad \Rightarrow \quad w = \frac{F l^3}{48EI}$$

APPLICATION OF CLAPEYRON ON BUCKLING PROBLEMS

First derived by John William Strutt Lord Rayleigh and known as Rayleigh Quotient.

The Right Honourable
The Lord Rayleigh
OM PC FRS



Rayleigh in 1904

Chancellor of the University of Cambridge

In office
1908–1919

Preceded by Spencer Cavendish, 8th Duke of Devonshire

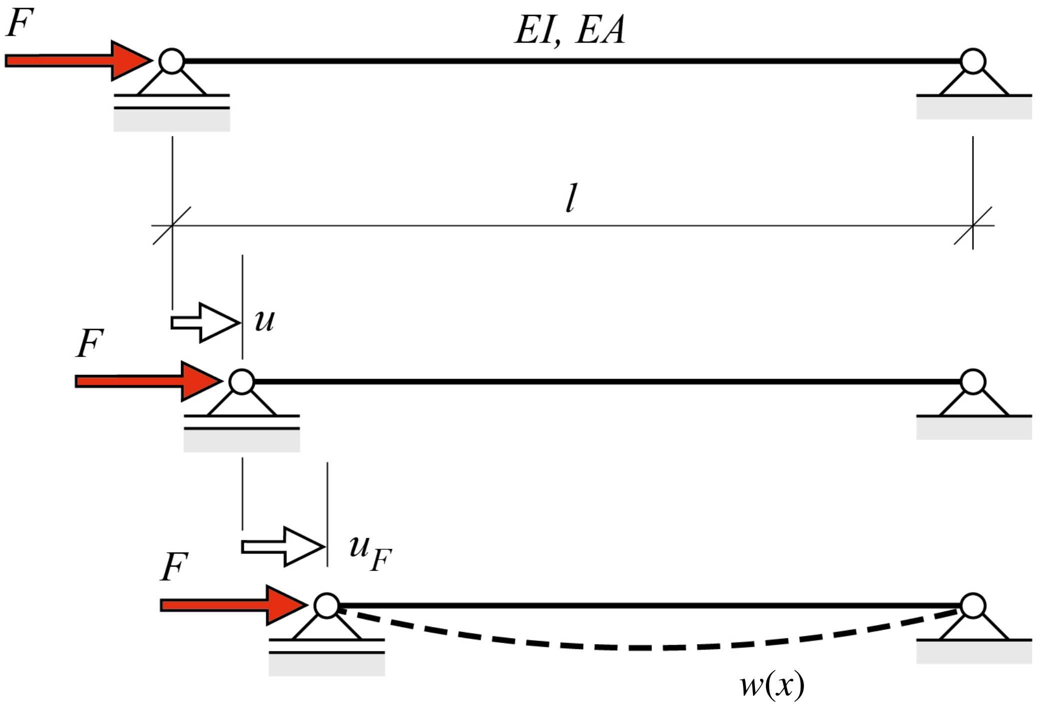
Succeeded by Arthur Balfour, 1st Earl of Balfour

39th President of the Royal Society

In office

Source : Wikipedia

Rayleigh's Quotient (visual method)

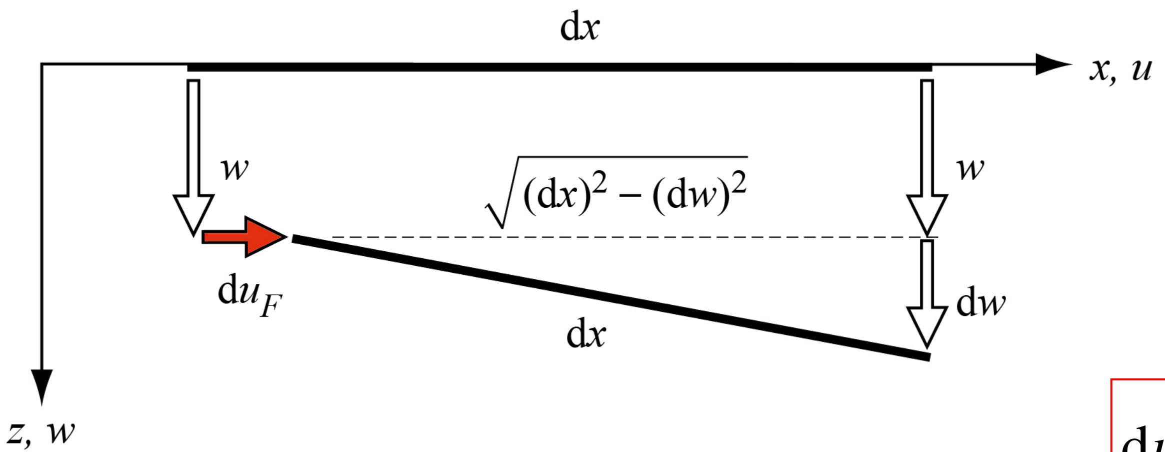


prior to buckling, only axial deformation

after buckling, axial and bending deformation

Find a relation between the buckling shape $w(x)$ and the extra horizontal displacement u_F during buckling.

HORIZONTAL DISPLACEMENT DURING BUCKLING



$$du_F = dx - \sqrt{(dx)^2 - (dw)^2} = \left(1 - \sqrt{1 - \left(\frac{dw}{dx}\right)^2}\right) dx$$

ELABORATE THIS ….

$$\mathrm{d}u_\mathrm{F} = \left(1 - \sqrt{1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2}\right)\mathrm{d}x \overset{Taylor}{\cong} \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x \quad \Leftrightarrow$$

$$u_\mathrm{F} = \int_0^l \mathrm{d}u_\mathrm{F} = \int_0^l \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

Relation between the buckling shape $w(x)$ and the extra horizontal displacement u_F during buckling.

Thus increase in work during buckling:

$$\Delta A = F\, u_F = F\int_0^l \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

So-called "eigen work", load is 100% available (axial load) and has a displacement u_F.

increase in strain energy:

$$\Delta E_v = \int_0^l \tfrac{1}{2} EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x$$

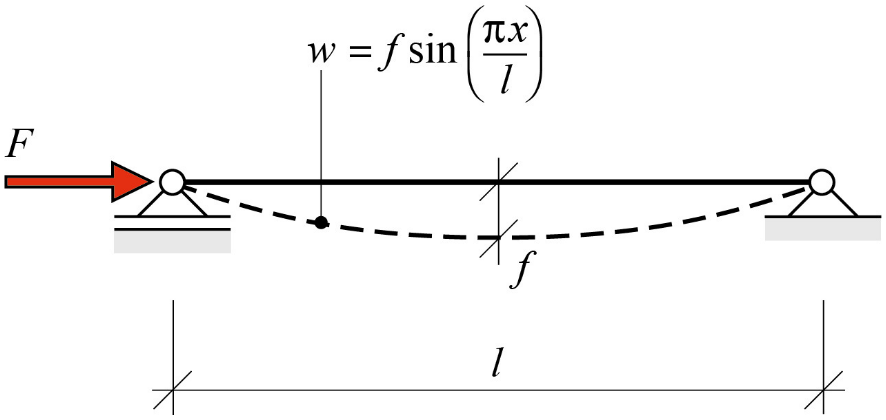
NOTE: curvature can be expressed in terms of $w(x)$ …

APPLY CLAPEYRON $(\Delta A = \Delta E)$

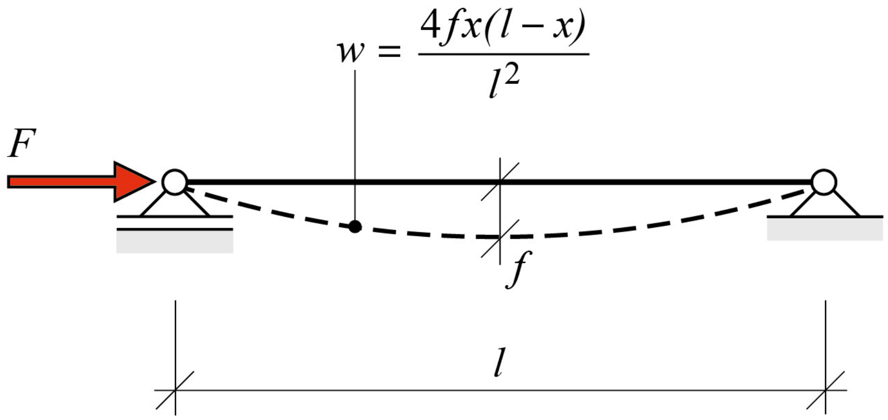
$$F_{\text{k-Rayleigh}} = \frac{\int\limits_0^l EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x}{\int\limits_0^l \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x}$$

Approximation of the buckling load based on an assumed buckling mode $w(x)$. This mode satisfies the kinematic boundary conditions and will produce a non-conservative value for the buckling load if it **is not** the actual buckling mode.

EXAMPLE



Assumption 1



Assumption 2

- Assume a kinematically admissible displacement field
- Elaborate the integrals in the expression and compute the Buckling Load …

RESULT

$$F_{\text{k-Rayleigh}} = \frac{\int_0^l EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x}{\int_0^l \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x} = two\ cases$$

Case 1:

$$\frac{f^2 EI \frac{\pi^2}{l^2} \int_0^l \left(\sin\frac{\pi x}{l}\right)^2 \mathrm{d}x}{f^2 \frac{\pi}{l} \int_0^l \left(\cos\frac{\pi x}{l}\right)^2 \mathrm{d}x} = \frac{\pi^2 EI}{l^2}$$

Solution for Assumption 1

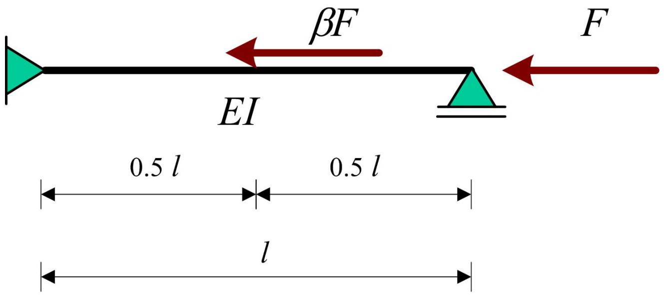
Case 2:

$$\frac{f^2 \int_0^l EI\left(\frac{64}{l^3}\right)^2 \mathrm{d}x}{f^2 \int_0^l \left(\frac{4(l-2x)}{l^2}\right)^2 \mathrm{d}x} = \frac{12EI}{l^2}$$

Solution for Assumption 2

NOTE: non-conservative

APPLICATION

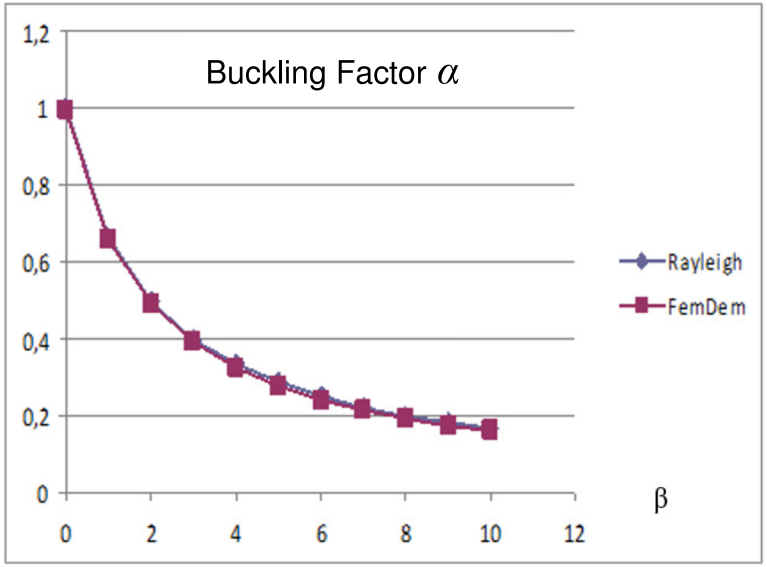


Assume:

$$w(x) = C \sin \frac{\pi x}{l}$$

$$F_k = \frac{2\pi^2 EI}{(\beta+2) l^2}$$

$$F_k = \alpha \frac{\pi^2 EI}{l^2} \quad \text{with:} \quad \alpha = \frac{2}{\beta+2}$$



STEPS

- Assume buckling mode (displacement field)

$$w(x) = C\sin\left(\frac{\pi x}{l}\right)$$

- Find horizontal displacement per load expressed in assumed displacement field

$$u(x) = -\int_0^x \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x = -\frac{C^2\pi}{4l^2}\left[l\cos(\frac{\pi x}{l})\sin(\frac{\pi x}{l}) + \pi x\right]$$

$$u(\tfrac{1}{2}l) = -\frac{C^2\pi^2}{8l}$$

$$u(l) = -\frac{C^2\pi^2}{4l}$$

- Find work done by the concentrated forces

$$\Delta A = (-\beta F)\times\frac{-C^2\pi^2}{8l} + (-F)\times\frac{-C^2\pi^2}{4l}$$

$$\Delta A = \frac{C^2 F\pi^2}{8l}(\beta+2)$$

- Find stored potential energy (strain energy) due to assumed displacement field

$$\Delta E_v = \int_0^l \tfrac{1}{2} EI\kappa^2 \mathrm{d}x = \int_0^l \tfrac{1}{2} EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = \frac{C^2\pi^4 EI}{4l^3}$$

- Apply Clapeyron's principle ($\Delta A = \Delta E$)

$$F_k = \frac{2\pi^2 EI}{(\beta+2)l^2} = \alpha\frac{\pi^2 EI}{l^2} \quad with: \quad \alpha = \frac{2}{\beta+2}$$