ANSWERS

Problem 1

a) The presented beam is a shear beam. The deformation is referred to as shear deformation and denoted with γ. The displacement field is the vertical displacement $w(x)$. Only loads in the direction of the displacement field can be modelled. In this case a distributed load $q(x)$. Discrete loads F can be used at interfaces or boundaries. The continuous model can be found with the three basic relations which will relate the displacements to the loads.

minimum set of required expressions (all quantities functions of x) :

kinematics : $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \gamma$

constitutive : $V = k\gamma$

equilibrium : $\dfrac{\mathrm{d}V}{\mathrm{d}x} = -q$ use sketch!

Substituting the three expressions results to the governing ODE:

$$k\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{\mathrm{d}k}{\mathrm{d}x}\frac{\mathrm{d}w}{\mathrm{d}x} = -q \quad \text{with:} \quad k = k_o e^{-bx}$$

With the general solution of this ODE and the boundary conditions, the displacement field can be obtained[1]. The boundary conditions for this problem can be written as:

$$w(0) = 0$$
$$V(l) = F$$

b) Find the general solution (basic math):

$$k_o e^{-bx}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - bk_o e^{-bx}\frac{\mathrm{d}w}{\mathrm{d}x} = 0 \quad \Leftrightarrow \quad w'' - bw' = 0 \quad \wedge \quad k_o e^{-bx} \neq 0$$

substitute $w = e^{\lambda x}$: $\lambda^2 - b\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0; \quad \lambda_2 = b;$

$$w = C_1 + C_2 e^{bx}; \quad w' = bC_2 e^{bx}; \quad V = k_o e^{-bx} bC_2 e^{bx} = bk_o C_2$$

Solve the integration constants with the boundary conditions.

$$w(0) = 0 \quad \Rightarrow \quad C_1 + C_2 = 0$$

$$V(l) = F \quad \Rightarrow \quad k_o bC_2 = F \quad \Rightarrow \quad C_2 = \frac{F}{k_o b}; \quad C_1 = -\frac{F}{k_o b}$$

The displacement field (continuous model) becomes:

$$w(x) = \frac{F}{k_o b}\left(e^{bx} - 1\right)$$

[1] Based upon the constant shear for this load case, with direct integration of the constitutive relation the displacement field can be obtained. This method is for this case of course also correct.

c) With this result, the displacement at B can be found:

$$w(l) = \frac{F}{k_o b}\left(e^{bl} - 1\right)$$

Note:
If only the displacement at B is modelled with a discrete spring the spring stiffness becomes:

$$k_{spring} = \frac{F}{w(l)} = \frac{F}{\frac{F}{k_o b}\left(e^{bl} - 1\right)} = \frac{k_o b}{\left(e^{bl} - 1\right)} \qquad \text{(with } l, b \text{ and } k_o \geq 0\text{)}$$

d) In order to model this beam with the Matrix Method an element is required with degrees of freedom in the direction of the continuous displacement field w. A local coordinate system is used with its local x-axis from node i to node j and a z-axis perpendicular to the x-axis. Each node i has one degree of freedom w_i. The shear element has therefore two degrees of freedom in the direction of the local z-axis. For a complete description an element stiffness matrix [2x2] is needed and an element load vector is needed in order to redistribute a possible continuous load on the element to nodal loads. The element load vector has two components which will contain the equivalent nodal loads for possible element loads. In this particular load case no element load is present.

e) The element stiffness can be found in several ways. From answer c) we already observed that in this case the beam can be replaced by a discrete spring with the presented spring stiffness. From this, the element stiffness matrix becomes:

$$K^{(e)} = \frac{k_o b}{\left(e^{bl} - 1\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Another alternative approach is to solve the homogeneous ODE for the boundary conditions:

$$w(0) = w_i$$
$$w(l) = w_j$$

With the solution the forces F_i and F_j at the element ends the follow holds:

$$F_i = -V(0)$$
$$F_j = V(l)$$

This results in:

$$F_i = \frac{bk_o}{\left(e^{bl} - 1\right)}\left(w_i - w_j\right)$$

$$F_j = \frac{bk_o}{\left(e^{bl} - 1\right)}\left(-w_i + w_j\right)$$

$$\begin{bmatrix} F_i \\ F_j \end{bmatrix} = \frac{bk_o}{\left(e^{bl} - 1\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} w_i \\ w_j \end{bmatrix}$$

stiffness matrix

Problem 2

a) see notes

b) Element (2) and (3) elements are linked to nodes which cannot move. These two elements can be described with rotational degrees of freedom only. Since element (1) cannot move horizontally, only the nodal rotation and displacement in z-direction is needed. The vertical displacement at node 2 will then only be modelled with element (1). The row and column of this dof will be “striked out” from the system since this vertical displacement will not occur.

The system degrees of freedom are: $w_1, \varphi_1, \varphi_2, \varphi_3, \varphi_4$

The constrained (supports) dof’s are $\varphi_1, \varphi_3, \varphi_4$.

The obtainable support reactions are: T_1, T_3, T_4

After reduction of the system only two unknown degrees of freedom have to be solved.

Element 1:

dof’s : $w_1, \varphi_1, w_2, \varphi_2$ (remove w_2 later)

$$K^{(1)} = \frac{EI}{l}\begin{bmatrix} \frac{12}{l^2} & -\frac{6}{l} & -\frac{12}{l^2} & -\frac{6}{l} \\ -\frac{6}{l} & 4 & \frac{6}{l} & 2 \\ -\frac{12}{l^2} & \frac{6}{l} & \frac{12}{l^2} & \frac{6}{l} \\ -\frac{6}{l} & 2 & \frac{6}{l} & 4 \end{bmatrix} = 500\begin{bmatrix} 0.75 & -1.50 & -0.75 & -1.50 \\ -1.50 & 4.00 & 1.50 & 2.00 \\ -0.75 & 1.50 & 0.75 & 1.50 \\ -1.50 & 2.00 & 1.50 & 4.00 \end{bmatrix}$$

$$f_{eq}^{(1)} = \begin{bmatrix} \frac{1}{2}ql \\ -\frac{1}{12}ql^2 \\ \frac{1}{2}ql \\ \frac{1}{12}ql^2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 12 \\ 8 \end{bmatrix}$$

Element 2:

dof’s : φ_2, φ_3

$$K^{(2)} = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = 1000\begin{bmatrix} 4.00 & 2.00 \\ 2.00 & 4.00 \end{bmatrix} = 500\begin{bmatrix} 8.00 & 4.00 \\ 4.00 & 8.00 \end{bmatrix}$$

Element 3:

dof’s : φ_2, φ_4

$$K^{(3)} = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = 1000\begin{bmatrix} 4.00 & 2.00 \\ 2.00 & 4.00 \end{bmatrix} = 500\begin{bmatrix} 8.00 & 4.00 \\ 4.00 & 8.00 \end{bmatrix}$$

c) To obtain the total system, the three elements have to be assembled in the global coordinate system. Since the local coordinate system of element (1) coincides with the global coordinate system and element (2) and (3) are described in rotational degrees of freedom only, no rotations are required for this step. The summation (assembling) is shown on the next page.

$$500\begin{bmatrix} 0.75 & -1.50 & -0.75 & -1.50 & & \\ -1.50 & 4.00 & 1.50 & 2.00 & & \\ -0.75 & 1.50 & 0.75 & 1.50 & & \\ -1.50 & 2.00 & 1.50 & 20.00 & 4.00 & 4.00 \\ & & & 4.00 & 16.0 & \\ & & & 4.00 & & 8.00 \end{bmatrix}\begin{bmatrix} w_1 \\ \varphi_1 \\ w_2 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} F+12 \\ T_1-8 \\ 12 \\ +8 \\ T_3 \\ T_4 \end{bmatrix}$$

d) From this system rows and columns of the constrained degrees of freedom can be "striked out". Row and column number three can be "striked out" since node 2 will not move. The right hand side of this third equation has no further meaning. This results in the reduced system:

$$\begin{bmatrix} 375 & -750 \\ -750 & 10000 \end{bmatrix}\begin{bmatrix} w_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} F+12 \\ 8 \end{bmatrix} = \begin{bmatrix} 72 \\ 8 \end{bmatrix}$$

From which the displacements can be found:

$$w_1 = 0.22776 \text{ m}; \quad \varphi_2 = 0.01788 \text{ rad};$$

The support reaction at node 1 (in this model) consists of a moment only. This can be found from the total system by substituting the known system displacement vector:

$$T_1 - 8 = -750 \times 0.22776 + 2000 \times 0 + 750 \times 0 + 1000 \times 0.01788 \quad \Leftrightarrow$$
$$T_1 = -144.94 \text{ kNm}$$

e) Since the horizontal and vertical displacements at node 2,3 and 4 and the horizontal displacement at node 1, have been removed from the model, a support reaction in any of these directions cannot be obtained directly from solving the system of equations. Without modelling the axial deformation in the reduced model it is not possible to find the complete normal force distribution. The moment and shear distribution for all elements however is complete.

Note: 1) Some of the reactions and axial forces however can be found based on equilibrium but this is not the standard solution strategy of the MatrixMethod.

2) An alternative reduction is to replace element (2) and (3) by a rotational spring with each a spring stiffness $\frac{4EI}{l}$. This results in only one beam-element with an additional rotational spring at node 2 resulting in exact the same reduced system.

Problem 3

a) See the lecture material.

b) The classical approach can be used. Since this arch is the symmetrical half of a full arch. The horizontal force H in the arch will be constant over the entire arch. So finding H will also give the answer for the horizontal force at the roller.

$$H = -\frac{\int_{arch} \frac{M^a z}{EI}\mathrm{dx}}{\int_{arch} \frac{z^2}{EI}\mathrm{dx} + \frac{l}{EA}} \quad \text{with:} \quad M^a = \tfrac{1}{2} q_o x(l-x); \quad z = -f \sin\frac{\pi x}{l} \quad \text{and} \quad EA = \infty$$

Elaborating this expression yields: (use the math tools from the formula sheet)

$$H = -\frac{\int_0^l \left(\tfrac{1}{2} q_o x(l-x)\right)\left(-f \sin\frac{\pi x}{l}\right)\mathrm{dx}}{\int_0^l \left(-f \sin\frac{\pi x}{l}\right)\mathrm{dx}} \quad \Leftrightarrow$$

$$H = -\frac{-\tfrac{1}{2} f q_o l \int_0^l x \sin\frac{\pi x}{l}\mathrm{dx} + \tfrac{1}{2} f q_o \int_0^l x^2 \sin\frac{\pi x}{l}\mathrm{dx}}{f^2 \int_0^l \sin^2\frac{\pi x}{l}\mathrm{dx}} \quad \Leftrightarrow$$

$$H = -\frac{-\tfrac{1}{2} f q_o l \frac{l^2}{\pi} + \tfrac{1}{2} f q_o \frac{l^3\left(\pi^2 - 4\right)}{\pi^3}}{f^2 \frac{l}{2}} = \frac{4 l^2 q_o}{\pi^3 f} = 275.2 \text{ kN}$$

c) With the given value of H the moment distribution in the arch can be found:

$$M(x) = M^a + Hz = \frac{1}{2} q_o x(l-x) + \frac{4l^2 q_o}{\pi^3 f}\left(-f \sin\frac{\pi x}{80}\right)$$

$$M(x) = \frac{5x(80-x)}{2} - \frac{128000}{\pi^3}\sin\frac{\pi x}{80}$$

d) At the roller a moment will occur of:

$$x = \tfrac{1}{2} l \quad M = 4000 - \frac{128000}{\pi^3} = -128.2 \text{ kNm}$$

The maximum field moment of 170.2 kNm will occur at x = 10.09 m. (not asked for)

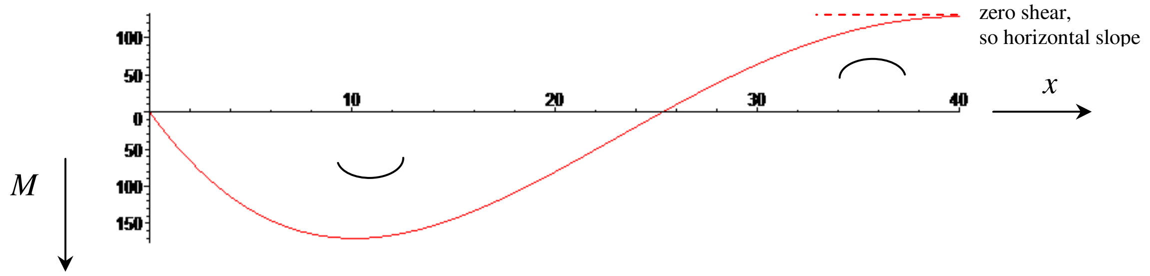

Figure 4 : Moment distribution in the arch.