Faculty
Civil Engineering and GeoSciences

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| Exam | **CIE4190** |
| | **Slender Structures** |
| Total number of pages | **8 pages (excl cover)** |
| Date and time | **JAN-25-2019** from **13:30-16:30** |
| Responsible lecturer | **J.W. Welleman** |

***Only the work / answers written on examination paper will be assessed, unless otherwise specified under 'Additional Information'.***

**Exam questions** (to be filled in by course examiner)

Total number of questions: 4

☒ **questions may differ in weight** *(the time mentioned is an indicator for the weight)*

**Use of tools and sources of information** (to be filled in by course examiner)

Not allowed:

- Mobile phone, smart Phone or devices with similar functions.
- Answers written with red pen or with pencils.
- Calculators with CAS and/or wifi/BT and/or PDF capabilities
- Tools and/or sources of information unless otherwise specified below.

Allowed:

☐ **books** ☐ **notes** ☐ **dictionaries** ☐ **syllabus**

☐ **formula sheets (see also below under 'additional information')** ☒ **calculators**

☐ **computer** ☐ **...**

☒ **scientific (graphical)calculator** ☒ **drawing material**

**Additional information** (if necessary to be filled in by the examiner)

- **Use for each problem a separate examination paper**
- **The question form contains fomula sheets which can be used.**
- **Students can take the question form home after the exam.**
- **No student leaves without delivering an exam paper with a name on it!**

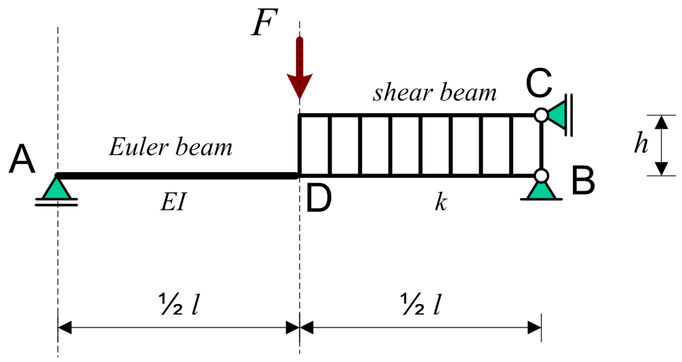
**Exam graded by:** (the marking period is 15 working days at most)

Every suspicion of fraud is reported to the Board of Examiners.

Mobile Phone OFF.

## Problem 1 : Matrix method ( 45 min )

In figure 1 a beam structure is presented. Part AD is a so-called Euler beam and part DBC is a shear beam. The beam is loaded at midspan with a concentrated load $F$ as indicated. The influence of axial deformation can be neglected.




Given : $l$ = 8 m; $h$ = 0.8 m; $F$ = 22 kN; $k$ = 1000 kN; $EI$ = 6400 kNm$^2$

*Figure* 1 : Beam structure

Questions will deal with the discrete solving technique also known as MatrixMethod.

**Questions:**

a) Respond to the following proposition:

*In general, a shear beam can be modelled with an ODE in terms of the transverse displacement field.*

b) Derive, based on the ODE or shape functions, the element stiffness matrix for the shear beam which is required in this problem.

c) In case the MatrixMethod is applied to model this structure, sketch the “model” with its degrees of freedom and all parameters needed in the description. Clearly label all unknowns in your model and specify the element definitions used.

d) Set up the system to solve the unknowns and present this in matrix notation. Clearly show all relevant intermediate steps involved.

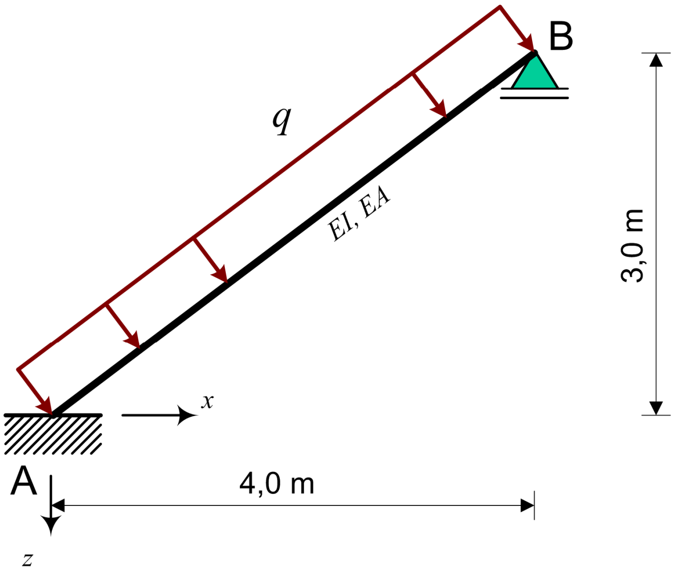
e) Solve all the unknowns and find the displacement of point D.

**Bonus**

f) Find the horizontal reaction force at B.

## Problem 2 : Basic cases ( 45 min )

A simple beam structure is presented in figure 2. The beam with only axial and bending deformation is supported at B with a horizontal roller. At A the beam is fully clamped. Any deformation due to shear can be neglected.




Given : $q = 80.8$ kN/m; $EA = 27000$ kN; $EI = 4000$ kNm$^2$

*Figure* 2 : Beam

**Questions:**

a) Derive the governing differential equations needed to model this structure and give the general solutions(s) for the displacement field(s) and clearly show the unknowns in your model. You are free to define your own coordinate system.

b) Specify all boundary conditions and use these to specify the equations needed to solve your unknowns.

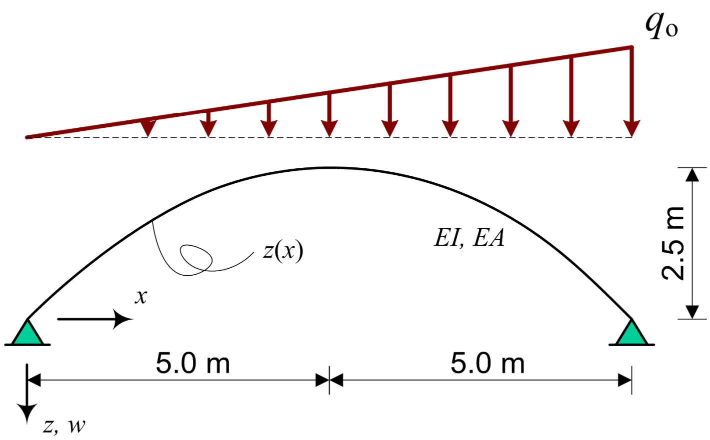
c) Solve the unknowns and find, based on this result, all support reactions and show how these reactions act on the beam.

d) Sketch the moment-, shear force and normal force distribution for the beam. In case you could not obtain results you may assume a shear force magnitude of 150 kN at B.

e) Find the horizontal displacement of the roller at B.

## Problem 3 : Arches ( 45 min )

A parabolic arch is presented in figure 3. The arch is loaded with a distributed load. The load distribution is linear as can be observed from figure 3. The load acts vertical and is expressed as load per unit length measured along the $x$-axis. The maximum load $q_o$ occurs at the right end of the span. The axial deformation of the arch will be neglected..




Given : $l$ = 10 m; $f$ = 2.5 m; $q_o$ = 50 kN/m; $EI$ = 25000 kNm$^2$

*Figure* 3 : Arch structure

**Questions:**

a) Describe in a few lines the load carrying capacity of this arch and explain the two principle techniques to solve the force distribution for arches. Clearly explain the limitations of the technique(s).

b) Find for the presented arch the expression for the moment distribution as a function of $x$. If needed use the formulas presented on the formula sheet which is attached to this exam. You are free to use any method you think is best suitable for this particular case.

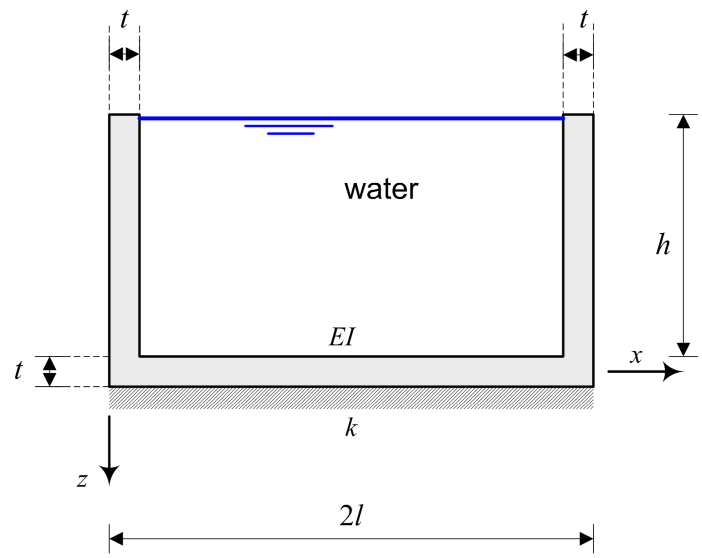
c) Sketch the moment distribution and include deformation symbols and values at characteristic points. Compare this moment distribution with the distribution which occurs for a straight beam with similar support conditions and bending stiffness. For this latter situation you are allowed to use a rough estimate of the moment distribution.

d) In case the support at B will have a prescribed horizontal displacement $u$. How could you find the moment distribution. Describe all relevant steps but do not solve this.

## Problem 4 : Beam on an elastic foundation ( 45 min )

A segment of an aqueduct with a water depth of 3.5 m is based on an elastic foundation with stiffness $k$ as indicated in figure 4. This (educational) structure has a constant wall thickness of 0.5 m and the specified bending stiffness $EI$. The concrete used has a density of 24 kN/m$^3$. The total width of the aqueduct is 20.5 m.

This problem focusses on the base slab of the aqueduct. For that we model a 1.0 m segment in longitudinal direction of the aqueduct with the presented cross section. The base slab will be modelled as a beam on an elastic foundation and the width of the aqueduct is sufficient to obtain at mid span a deflection which is equal to the particular solution. Any influence of the possible axial deformation is neglected in this problem.




Given : $l = 10.25$ m; $h = 3.5$ m; $t = 0.5$ m; $EI = 1500 \times 10^3$ kNm$^2$; $k = \dfrac{\pi^4}{64} EI$ kN/m$^2$

*Figure* 4 : Aqueduct on an elastic foundation

**Questions:**

a) Model the base slab of this structure and clearly specify all loads on it.

b) Give the general solution for the base slab which you will use to find the force distribution in the slab. Clearly explain possible simplifications.

c) Specify all boundary conditions needed to solve the unknowns.

d) Solve the unknowns and sketch the displacement field and specify relevant values at characteristic points.

e) Sketch the shear force distribution in the slab. Specify the maximum value of the shear force at its location in the graph.

# FORMULAS

**Temperature:**

$$N = EA(\varepsilon - \varepsilon^T); \qquad M = EI(\kappa - \kappa^T)$$

$$\varepsilon^T = \frac{\alpha}{A}\int_A T(x,z)\mathrm{d}A; \quad \kappa^T = \frac{\alpha}{I}\int_A zT(x,z)\mathrm{d}A$$

**Stress distributions in beams:**

$$\sigma(z) = \frac{N}{A} + \frac{Mz}{I}$$

$$\tau(z) = \frac{6V\left(\frac{1}{4}h^2 - z^2\right)}{bh^3}$$ rectangular crosssections

**Arch:**

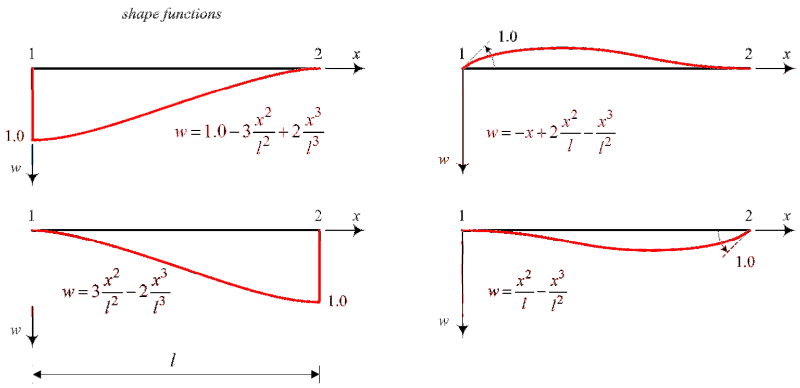
$$H = -\frac{\displaystyle\int_{arch} \frac{M^a z}{EI}\mathrm{d}x}{\displaystyle\int_{arch} \frac{z^2}{EI}\mathrm{d}x + \frac{l}{EA}}$$

**Cable:**

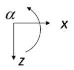
$$H^2 = \frac{q^2 l^3}{24\Delta};$$

$$f = \sqrt{\frac{3}{8} l\,\Delta} \text{ or } \Delta = \frac{8f^2}{3l}$$

**Matrix Methods**



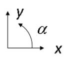

$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$




$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$\bar{f} = R\,f$ and $f = R^T\bar{f}$

$\bar{u} = R\,u$ and $u = R^T\bar{u}$




$$R = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

**Math tools:**

$$y = Ae^{-\beta x}\sin(\beta x + \omega)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{2}\,\beta A e^{-\beta x}\sin\left(\beta x + \omega - \tfrac{1}{4}\pi\right)$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\beta^2 A e^{-\beta x}\sin\left(\beta x + \omega - \tfrac{1}{2}\pi\right)$$

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -2\sqrt{2}\beta^3 A e^{-\beta x}\sin\left(\beta x + \omega - \tfrac{3}{4}\pi\right)$$

$$\int_0^l x^2(l-x)^2(l+x)\,dx = \frac{l^6}{20}$$

$$\int\sqrt{1+x^2}\mathrm{d}x \approx \int\left[1 + \frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]\mathrm{d}x$$

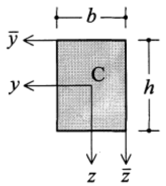
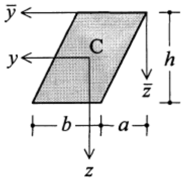
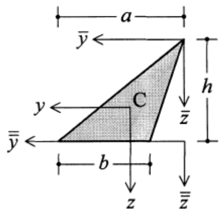
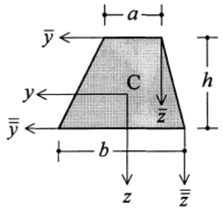
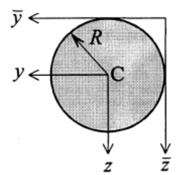
$$\int_0^a \sin^2\frac{\pi x}{a}\mathrm{d}x = \frac{a}{2} \qquad \int_0^a x\sin\frac{\pi x}{a}\mathrm{d}x = \frac{a^2}{\pi}$$

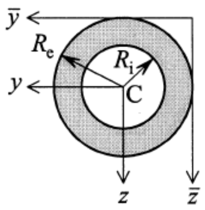
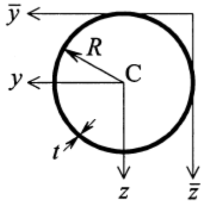
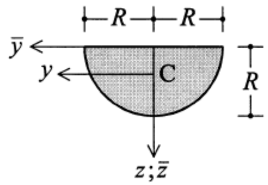
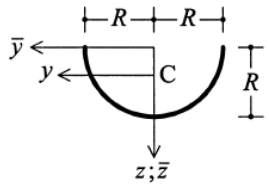
$$\int_0^a x^2\sin\frac{\pi x}{a}\mathrm{d}x = \frac{a^3\left(\pi^2 - 4\right)}{\pi^3}$$

$$\int_0^l \left(\frac{x(l-x)}{l^2}\right)^2 dx = \frac{l}{30}$$

*catenary solution* :

$$z = -\frac{H}{q}\cosh\left(-\frac{qx}{H} + C_1\right) + C_2$$

| Figure | Area, coordinates centroid C | Second moments of area: centroidal | Second moments of area: other |
|---|---|---|---|
|  | Rectangle<br>$A = bh$<br>$\bar{y}_C = \frac{1}{2}b$<br>$\bar{z}_C = \frac{1}{2}h$ | $I_{yy} = \frac{1}{12}b^3h$<br>$I_{zz} = \frac{1}{12}bh^3$<br>$I_{yz} = 0$ | $I_{\bar{y}\bar{y}} = \frac{1}{3}b^3h$<br>$I_{\bar{z}\bar{z}} = \frac{1}{3}bh^3$<br>$I_{\bar{y}\bar{z}} = \frac{1}{4}b^2h^2$ |
|  | Parallelogram<br>$A = bh$<br>$\bar{y}_C = \frac{1}{2}(a+b)$<br>$\bar{z}_C = \frac{1}{2}h$ | $I_{yy} = \frac{1}{12}(a^2+b^2)bh$<br>$I_{zz} = \frac{1}{12}bh^3$<br>$I_{yz} = \frac{1}{12}abh^2$ | $I_{\bar{z}\bar{z}} = \frac{1}{3}bh^3$ |
|  | Triangle<br>$A = \frac{1}{2}bh$<br>$\bar{y}_C = \frac{1}{3}(2a-b)$<br>$\bar{z}_C = \frac{2}{3}h$ | $I_{yy} = \frac{1}{36}(a^2-ab+b^2)bh$<br>$I_{zz} = \frac{1}{36}bh^3$<br>$I_{yz} = \frac{1}{72}(2a-b)bh^2$ | $I_{\bar{z}\bar{z}} = \frac{1}{4}bh^3$<br>$I_{\bar{y}\bar{z}} = \frac{1}{8}(2a-b)bh^2$<br>$I_{\bar{\bar{z}}\bar{\bar{z}}} = \frac{1}{12}bh^3$ |
|  | Trapezium<br>$A = \frac{1}{2}(a+b)h$<br>$\bar{z}_C = \frac{1}{3}\frac{a+2b}{a+b}h$ | $I_{zz} = \frac{1}{36}\frac{a^2+4ab+b^2}{a+b}h^3$ | $I_{\bar{z}\bar{z}} = \frac{1}{12}(a+3b)h^3$<br>$I_{\bar{\bar{z}}\bar{\bar{z}}} = \frac{1}{12}(3a+b)h^3$ |
|  | Circle<br>$A = \pi R^2$ | $I_{yy} = I_{zz} = \frac{1}{4}\pi R^4$<br>$I_{yz} = 0$<br>$I_p = \frac{1}{2}\pi R^4$ | $I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{5}{4}\pi R^4$<br>$I_{\bar{y}\bar{z}} = \pi R^4$ |

| Figure | Area, coordinates centroid C | Second moments of area: centroidal | Second moments of area: other |
|---|---|---|---|
|  | Thick-walled ring<br>$A = \pi(R_e^2 - R_i^2)$ | $I_{yy} = I_{zz} = \frac{1}{4}\pi(R_e^4 - R_i^4)$<br>$I_{yz} = 0$<br>$I_p = \frac{1}{2}\pi(R_e^4 - R_i^4)$ | |
|  | Thin-walled ring<br>$A = 2\pi Rt$ | $I_{yy} = I_{zz} = \pi R^3 t$<br>$I_{yz} = 0$<br>$I_p = 2\pi R^3 t$ | $I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = 3\pi R^3 t$ |
|  | Semicircle<br>$A = \frac{1}{2}\pi R^2$<br>$\bar{y}_C = 0$<br>$\bar{z}_C = \frac{4}{3\pi}R$<br>$= 0.424R$ | $I_{yy} = \frac{1}{8}\pi R^4 = 0.393R^4$<br>$I_{zz} = (\frac{\pi}{8} - \frac{8}{9\pi})R^4$<br>$= 0.110R^4$<br>$I_{yz} = 0$ | $I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{1}{8}\pi R^4$<br>$I_{\bar{y}\bar{z}} = 0$ |
|  | Semicircular ring<br>$A = \pi Rt$<br>$\bar{y}_C = 0$<br>$\bar{z}_C = \frac{2}{\pi}R$<br>$= 0.637R$ | $I_{yy} = \frac{1}{2}\pi R^3 t$<br>$I_{zz} = (\frac{\pi}{2} - \frac{4}{\pi})R^3 t$<br>$= 0.298R^3 t$<br>$I_{yz} = 0$ | $I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{1}{2}\pi R^3 t$<br>$I_{\bar{y}\bar{z}} = 0$ |

*forget-me-nots*

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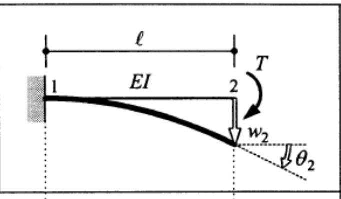
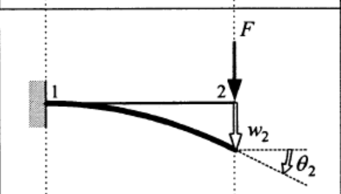
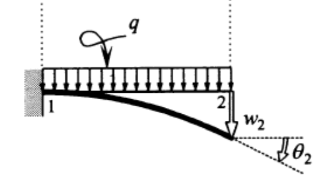
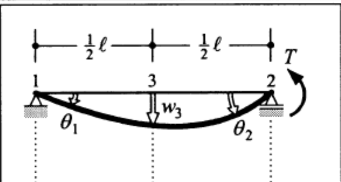
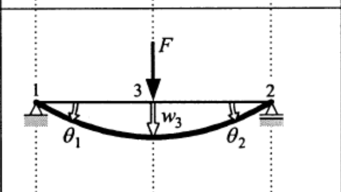
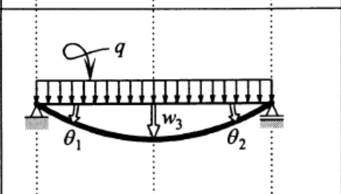
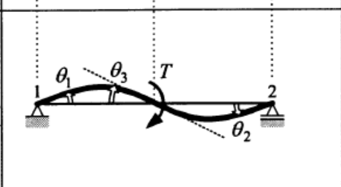
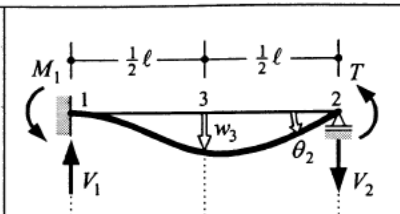
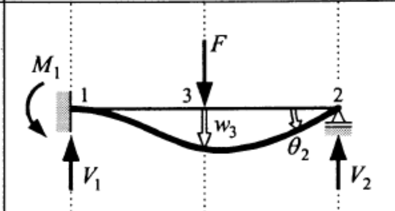
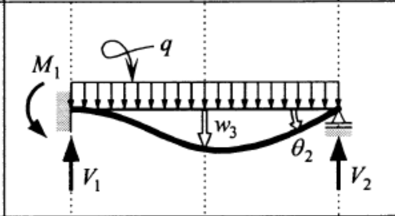
|---|---|---|
| (1) |  | $\theta_2 = \dfrac{T\ell}{EI}$; $w_2 = \dfrac{T\ell^2}{2EI}$ |
| (2) |  | $\theta_2 = \dfrac{F\ell^2}{2EI}$; $w_2 = \dfrac{F\ell^3}{3EI}$ |
| (3) |  | $\theta_2 = \dfrac{q\ell^3}{6EI}$; $w_2 = \dfrac{q\ell^4}{8EI}$ |

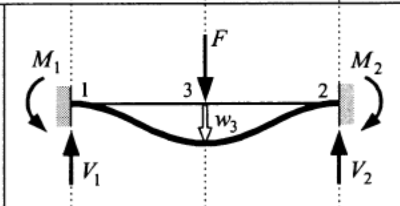
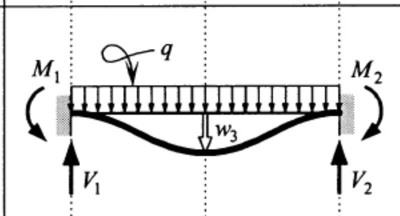
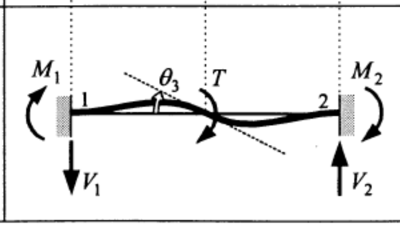
*simply supported beam (statically determinate)*

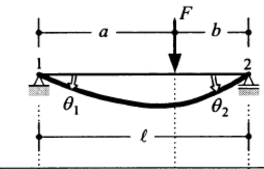
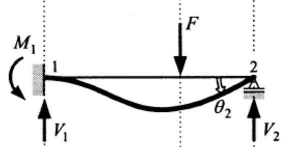
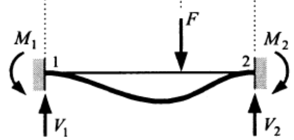
| | | |
|---|---|---|
| (4) |  | $\theta_1 = \dfrac{1}{6}\dfrac{T\ell}{EI}$; $\theta_2 = \dfrac{1}{3}\dfrac{T\ell}{EI}$; $w_3 = \dfrac{1}{16}\dfrac{T\ell^2}{EI}$ |
| (5) |  | $\theta_1 = \theta_2 = \dfrac{1}{16}\dfrac{F\ell^2}{EI}$; $w_3 = \dfrac{1}{48}\dfrac{F\ell^3}{EI}$ |
| (6) |  | $\theta_1 = \theta_2 = \dfrac{1}{24}\dfrac{q\ell^3}{EI}$; $w_3 = \dfrac{5}{384}\dfrac{q\ell^4}{EI}$ |
| (a) |  | $\theta_1 = \theta_2 = \dfrac{1}{24}\dfrac{T\ell}{EI}$; $\theta_3 = \dfrac{1}{12}\dfrac{T\ell}{EI}$; $w_3 = 0$ |

*statically indeterminate beam (one fixed end)*

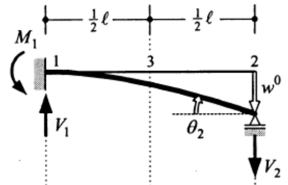
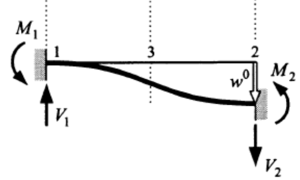
| | | |
|---|---|---|
| (7) |  | $\theta_2 = \dfrac{1}{4}\dfrac{T\ell}{EI}$; $w_3 = \dfrac{1}{32}\dfrac{T\ell^2}{EI}$<br>$M_1 = \dfrac{1}{2}T$; $V_1 = V_2 = \dfrac{3}{2}\dfrac{T}{\ell}$ |
| (8) |  | $\theta_2 = \dfrac{1}{32}\dfrac{F\ell^2}{EI}$; $w_3 = \dfrac{7}{768}\dfrac{F\ell^3}{EI}$<br>$M_1 = \dfrac{3}{16}F\ell$; $V_1 = \dfrac{11}{16}F$; $V_2 = \dfrac{5}{16}F$ |
| (9) |  | $\theta_2 = \dfrac{1}{48}\dfrac{q\ell^3}{EI}$; $w_3 = \dfrac{1}{192}\dfrac{q\ell^4}{EI}$<br>$M_1 = \dfrac{1}{8}q\ell^2$; $V_1 = \dfrac{5}{8}q\ell$; $V_2 = \dfrac{3}{8}q\ell$ |

*statically indeterminate beam (two fixed ends)*

| | | |
|---|---|---|
| (10) |  | $w_3 = \dfrac{1}{192}\dfrac{F\ell^3}{EI}$<br>$M_1 = M_2 = \dfrac{1}{8}F\ell$; $V_1 = V_2 = \dfrac{1}{2}F$ |
| (11) |  | $w_3 = \dfrac{1}{384}\dfrac{q\ell^4}{EI}$<br>$M_1 = M_2 = \dfrac{1}{12}q\ell^2$; $V_1 = V_2 = \dfrac{1}{2}q\ell$ |
| (b) |  | $\theta_3 = \dfrac{1}{16}\dfrac{T\ell}{EI}$; $w_3 = 0$<br>$M_1 = M_2 = \dfrac{1}{4}T$; $V_1 = V_2 = \dfrac{3}{2}\dfrac{T}{\ell}$ |

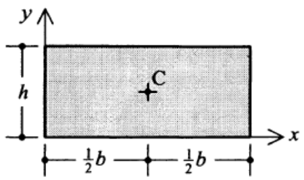
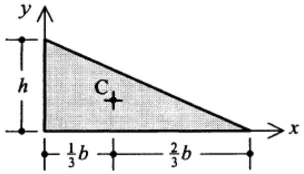
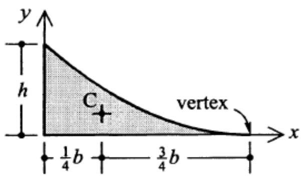
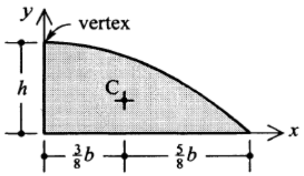
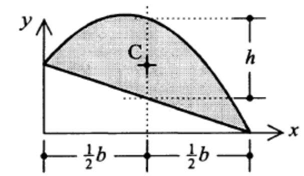
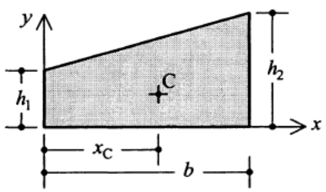
| | | |
|---|---|---|
| (c) |  | $\theta_1 = \dfrac{Fab(\ell+b)}{6EI\ell} = \dfrac{F\ell^2}{6EI}\left(2\dfrac{a}{\ell} - 3\dfrac{a^2}{\ell^2} + \dfrac{a^3}{\ell^3}\right)$<br>$\theta_2 = \dfrac{Fab(\ell+a)}{6EI\ell} = \dfrac{F\ell^2}{6EI}\left(\dfrac{a}{\ell} - \dfrac{a^3}{\ell^3}\right)$ |
| (d) |  | $M_1 = \dfrac{Fb(\ell^2-b^2)}{2\ell^2} = F\ell\left(\dfrac{a}{\ell} - \dfrac{3}{2}\dfrac{a^2}{\ell^2} + \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$<br>$V_1 = \dfrac{Fb(3\ell^2-b^2)}{2\ell^3} = F\left(1 - \dfrac{3}{2}\dfrac{a^2}{\ell^2} + \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$<br>$V_2 = \dfrac{Fa^2(3\ell-a)}{2\ell^3} = F\left(\dfrac{3}{2}\dfrac{a^2}{\ell^2} - \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$<br>$\theta_2 = \dfrac{Fa^2b}{4EI\ell} = \dfrac{F\ell^2}{4EI}\left(\dfrac{a^2}{\ell^2} - \dfrac{a^3}{\ell^3}\right)$ |
| (e) |  | $M_1 = \dfrac{Fab^2}{\ell^2} = F\ell\left(\dfrac{a}{\ell} - 2\dfrac{a^2}{\ell^2} + \dfrac{a^3}{\ell^3}\right)$<br>$V_1 = \dfrac{Fb^2(\ell+2a)}{\ell^3} = F\left(1 - 3\dfrac{a^2}{\ell^2} + 2\dfrac{a^3}{\ell^3}\right)$<br>$M_2 = \dfrac{Fa^2b}{\ell^2} = F\ell\left(\dfrac{a^2}{\ell^2} - \dfrac{a^3}{\ell^3}\right)$<br>$V_2 = \dfrac{Fa^2(\ell+2b)}{\ell^3} = F\ell\left(3\dfrac{a^2}{\ell^2} - 2\dfrac{a^3}{\ell^3}\right)$ |

*support reactions and rotations at the beam ends*

| | | |
|---|---|---|
| (f) |  | $M_1 = \dfrac{3EI}{\ell^2}w^0;\quad V_1 = V_2 = \dfrac{3EI}{\ell^3}w^0$<br>$\theta_2 = \dfrac{3}{2}\dfrac{w^0}{\ell}$<br>$\theta_3 = \dfrac{9}{8}\dfrac{w^0}{\ell};\quad w_3 = \dfrac{5}{16}w^0$ |
| (g) |  | $M_1 = M_2 = \dfrac{6EI}{\ell^2}w^0;\quad V_1 = V_2 = \dfrac{12EI}{\ell^3}w^0$<br>$\theta_3 = \dfrac{3}{2}\dfrac{w^0}{\ell};\quad w_3 = \dfrac{1}{2}w^0$ |

*settlements*

*properties of plane figures to be used for the moment-area theorems*

| | | |
|---|---|---|
| (1) |  | rectangle: $y = h$<br>$A = bh$<br>$x_C = \frac{1}{2}b$ |
| (2) |  | triangle: $y = h\left\{1 - \dfrac{x}{b}\right\}$<br>$A = \frac{1}{2}bh$<br>$x_C = \frac{1}{3}b$ |
| (3) |  | parabola: $y = h\left\{1 - \dfrac{x}{b}\right\}^2$<br>$A = \frac{1}{3}bh$<br>$x_C = \frac{1}{4}b$ |
| (4) |  | parabola: $y = h\left\{1 - \left(\dfrac{x}{b}\right)^2\right\}$<br>$A = \frac{2}{3}bh$<br>$x_C = \frac{3}{8}b$ |
| (5) |  | parabola:<br>$A = \frac{2}{3}bh$<br>$x_C = \frac{1}{2}b$ |
| (6) |  | trapezium: $y = h_1 + (h_2 - h_1)\dfrac{x}{b}$<br>$A = \frac{1}{2}b(h_1 + h_2)$<br>$x_C = \frac{1}{3}b\dfrac{h_1 + 2h_2}{h_1 + h_2}$ |