ANSWERS

Problem 1

a) Navier-beam is referred also to as Euler Bernoulli beam, so answer is beam 2

b) part 1 :

Beam 1 is a beam with shear and bending deformation and is referred to as a so-called Timoshenko beam. Beam 2 is a beam with only bending deformations and is referred to as an Euler-Bernoulli beam.

minimum set of required expressions:

beam 1

kinematics : $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \gamma - \varphi$

constitutive : $V = GA\gamma$, $\quad M = EI\dfrac{\mathrm{d}\varphi}{\mathrm{d}x}$

equilibrium : $\dfrac{\mathrm{d}V}{\mathrm{d}x} = -q$, $\quad \dfrac{\mathrm{d}M}{\mathrm{d}x} = V$ use sketch!

beam 2

kinematics : $\dfrac{\mathrm{d}w}{\mathrm{d}x} = -\varphi$

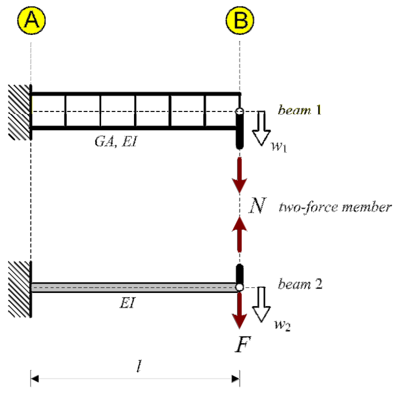
constitutive : $M = EI\dfrac{\mathrm{d}\varphi}{\mathrm{d}x}$

equilibrium : $\dfrac{\mathrm{d}V}{\mathrm{d}x} = -q$, $\quad \dfrac{\mathrm{d}M}{\mathrm{d}x} = V$ refer to previous sketch

part 2 :

solution technique

This problem is simply statically indeterminate. The normal force N in the rigid link is taken as the static unknown. For both beams the moment and shear distribution can be expressed in terms of N. At B both beams will have the same displacement w. With this equation a solution for N can be found. The displacement field per beam can be found based upon integration:



Beam 1: Rotation obtained from the moment distribution:

$$\varphi_1 = \int_0^l \frac{M_1}{EI}\,\mathrm{d}x \quad \text{and} \quad w_1 = \int_0^l \left(\frac{V_1}{GA} - \varphi_1\right)\mathrm{d}x$$

Beam 2: Almost the same :

$$\varphi_2 = \int_0^l \frac{M_2}{EI}\,\mathrm{d}x \quad \text{and} \quad w_1 = \int_0^l \left(-\varphi_2\right)\mathrm{d}x$$

c) Assume an unknown normal force N in the rigid link:

$$V_1(x) = N \qquad V_2(x) = -N + F$$
$$M_1(x) = -N\left(l - x\right) \qquad M_2(x) = -(-N + F)\left(l - x\right)$$

$$\varphi_1 = \int_0^x \frac{M_1}{EI}\mathrm{d}x = \frac{1}{EI}\int_0^x -N(l-x)\mathrm{d}x = \frac{\left(-Nlx + \frac{1}{2}Nx^2\right)}{EI} + C_1$$

With zero rotation at A the integration constant must be zero.

$$w_1 = \int_0^x \frac{N}{GA} - \frac{\left(-Nlx + \frac{1}{2}Nx^2\right)}{EI}\mathrm{d}x = \frac{Nx}{GA} - \frac{\left(-\frac{1}{2}Nlx^2 + \frac{1}{6}Nx^3\right)}{EI} + C_2$$

With zero displacement at A the integration constant must be zero.
The displacement of beam 1 at B becomes:

$$w_1(l) = \frac{Nl}{GA} + \frac{Nl^3}{3EI}$$ (this is a well-known expression which was found in exercises)

In the same way the displacement of beam 2 at B can be found:

$$w_2(l) = \frac{\left(F - N\right)l^3}{3EI}$$ (this is a well-known expression)

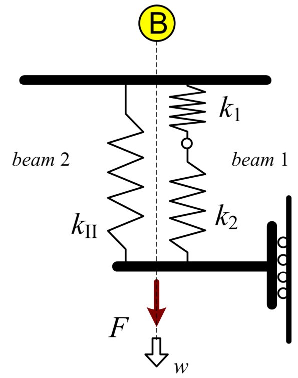
Since these two displacements must be identical (rigid link) the expression for N yields:

$$\frac{Nl}{GA} + \frac{Nl^3}{3EI} = \frac{\left(F - N\right)l^3}{3EI} \quad \Leftrightarrow$$
$$N\left(\frac{l}{GA} + \frac{2l^3}{3EI}\right) = \frac{Fl^3}{3EI} \quad \Leftrightarrow$$
$$N\left(\frac{3EIl + 2GAl^3}{3GAEI}\right) = \frac{Fl^3}{3EI} \quad \Leftrightarrow$$
$$N = \frac{3GAEI}{3EIl + 2GAl^3}\frac{Fl^3}{3EI} = \frac{Fl^2GA}{3EI + 2l^2GA}$$

d) With this result the displacement at B can be found:

$$w_1(l) = \frac{Nl}{GA} + \frac{Nl^3}{3EI} = \frac{Fl^2GA}{3EI + 2l^2GA} \times \frac{l}{GA} + \frac{Fl^2GA}{3EI + 2l^2GA} \times \frac{l^3}{3EI} \quad \Leftrightarrow$$
$$w_1(l) = \frac{Fl^3}{3EI + 2l^2GA} + \frac{Fl^5\left(GA/3EI\right)}{3EI + 2l^2GA} = \frac{3EI + l^2GA}{3EI + 2l^2GA} \times \frac{Fl^3}{3EI}$$

e) This system is a parallel system of two beams but the Timoshenko-beam can be regarded as a serial system for this load case.



beam 1 is a serial system of k_1 and k_2 resulting in a stiffness:

$$k_I = \frac{k_1 k_2}{k_1 + k_2} \quad \text{with} \quad k_1 = \frac{GA}{l} \text{ and } k_2 = \frac{3EI}{l^3}$$

beam 2 is a standard case with:

$$k_{II} = \frac{3EI}{l^3}$$

The resulting stiffness k of the parallel system yields:

$$k = k_I + k_{II} = \frac{3EI + 2l^2 GA}{3EI + l^2 GA} \times \frac{3EI}{l^3}$$

The displacement due to a load F then becomes:

$$w = \frac{F}{k} = \frac{3EI + l^2 GA}{3EI + 2l^2 GA} \times \frac{Fl^3}{3EI}$$

This result is identical to the previous found result.

Problem 2

a) see notes

b) elements 3,4 and 6 are loaded with element loads. Positive element loads act in the positive local z-direction. Element loads on (3) and (4) act in the local z-direction. For element (6) the local z-coordinate is opposite to the load on this element so a negative load should be applied on this element. The input for the element load thus becomes:

element 3 : q = 6 kN/m2
element 4 : q = 6 kN/m2
element 6 : q = -6 kN/m2

c) The system load vector is the right hand side in the system and has 15 components. Nodal loads and equivalent loads from element loads are assembled in this vector together with the unknown support reactions. In order to find the system load vector all element loads have to be assembled to the system vector in global coordinates. For this the transpose element rotation matrix is used. The element load vectors are:

$$f^{(e)} = \begin{bmatrix} 0 \\ \frac{1}{2}ql \\ -\frac{1}{12}ql^2 \\ 0 \\ \frac{1}{2}ql \\ \frac{1}{12}ql^2 \end{bmatrix} \quad f^{(4)} = \begin{bmatrix} 0 \\ 12 \\ -8 \\ 0 \\ 12 \\ 8 \end{bmatrix} \quad f^{(3)} = \begin{bmatrix} 0 \\ 12 \\ -8 \\ 0 \\ 12 \\ 8 \end{bmatrix} \quad f^{(6)} = \begin{bmatrix} 0 \\ -15 \\ 12.5 \\ 0 \\ -15 \\ -12.5 \end{bmatrix}$$

Only element (6) requires a rotation since the local coordinate system is not the same as the global coordinate system. Element (6) has an angle with the global system of 143.13 degrees. The contribution for element (6) to the system load vector becomes:

$$f^e = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2}ql \\ -\frac{1}{12}ql^2 \\ 0 \\ \frac{1}{2}ql \\ \frac{1}{12}ql^2 \end{bmatrix} \quad \text{with:} \quad \begin{aligned} s &= \sin(143.13) = 0.6; \\ c &= \cos(143.13) = -0.8; \end{aligned}$$

For element (6) this results in the following global force contributions:

$$f^6 = \begin{bmatrix} -0.8 & 0.6 & 0 & 0 & 0 & 0 \\ -0.6 & -0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.8 & 0.6 & 0 \\ 0 & 0 & 0 & -0.6 & -0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -15 \\ 12.5 \\ 0 \\ -15 \\ -12.5 \end{bmatrix} = \begin{bmatrix} -9 \\ 12 \\ 12.5 \\ -9 \\ 12 \\ -12.5 \end{bmatrix}$$

Before assembling element data, the degrees of freedom per element are needed (dof's numbered from node 1..5 in order of x,z and φ):

$$\begin{array}{ll} (1) = 1,2,3,4,5,6 & (4) = 1,2,3,7,8,9 \\ (2) = 4,5,6,13,14,15 & (5) = 7,8,9,10,11,12 \\ (3) = 4,5,6,10,11,12 & (6) = 10,11,12,13,14,15 \end{array}$$

The total system load vector f then becomes:

$$f = \begin{bmatrix} R_{1x} \\ R_{1z} + 12 \\ -8 \\ 0 \\ 12 \\ -8 \\ 0 \\ 12 \\ 8 \\ R_{4x} - 9 \\ 12 + 12 \\ 8 + 12.5 \\ -9 \\ 12 \\ -12.5 \end{bmatrix}$$

d) The row k_{10}^{T} in the stiffness matrix which belongs to the horizontal degree of freedom at node 4 (dof 10) is given. Multiply this row with the given system displacements vector u. This will result in the system load value of this degree of freedom in the system load vector from which the unknown support reaction can be found.

$$k_{10}^{T} u = R_{4x} - 9$$

Elaborating this expression results in:

$$-30.03182 = R_{x4} - 9 \quad \Leftrightarrow \quad R_{x4} = -21.03\,\text{kN} \quad (\leftarrow)$$

From moment equilibrium around point 1 of the entire structure, the horizontal support reaction at 4 can be found which results in a value of -21 kN. The difference in answer is due to the precision of the given displacement and stiffness values.

e) The k[10,4] is the result of the stiffness contribution $-EA/l$ = 15000/4 = -3750 from element (3). The k[10,9] is the result of the stiffness contribution $6EI/l^2$ = 18000/9 = 2000 from element (5). Only element (5) is connected to both dof 9 (φ_3) and dof 10 (u_4). Turn your head 90 degrees anti clock wise to use the local coordinates of element (5) and see which matrix element you need to use … indeed third row and 5th element.

Support these answers with a small sketch of the element matrix components and the connected dof's.

Problem 3

a) Superposition is not possible since for each load a different H will be found.

b) With the given exact value of H the position of the cable at the point of application of the load can simply be found based upon equilibrium:

$$Hz - \frac{Fa(l-a)}{l} = 0 \quad \Leftrightarrow \quad z = \frac{Fa(l-a)}{Hl}$$

with: $H = \dfrac{2(1+\beta)}{\sqrt{-8a^2\beta - 4a^2\beta^2 + 8al\beta + 4al\beta^2 + 4l^2\beta^2 + 4l^2\beta^3 + l^2\beta^4}} \dfrac{Fa(l-a)}{l}$

This results in the exact solution of z:

$$z = \frac{\sqrt{-8a^2\beta - 4a^2\beta^2 + 8al\beta + 4al\beta^2 + 4l^2\beta^2 + 4l^2\beta^3 + l^2\beta^4}}{2(1+\beta)}$$

c) Use the previously found expression for z based upon equilibrium:

$$z = \frac{Fa(l-a)}{Hl} \quad \text{and:} \quad (1+\beta)l = \sqrt{z^2 + a^2} + \sqrt{z^2 + (l-a)^2}$$

The total length of the cable in its equilibrium position is given which is shown in the last expression. Substituting the expression for z in the first equilibrium condition will result in the "exact" value of H.

d) The length of the cable can be approximated with a Taylor approximation:

$$(1+\beta)l = a\sqrt{1+\left(\frac{z}{a}\right)^2} + (l-a)\sqrt{1+\left(\frac{z}{l-a}\right)^2} \quad \Leftrightarrow$$

$$(1+\beta)l \approx a\left(1+\tfrac{1}{2}\left(\frac{z}{a}\right)^2\right) + (l-a)\left(1+\tfrac{1}{2}\left(\frac{z}{l-a}\right)^2\right) \quad \Leftrightarrow$$

$$l + \beta l \approx a + \tfrac{1}{2}\frac{z^2}{a} + l - a + \tfrac{1}{2}\left(\frac{z^2}{l-a}\right) \quad \Rightarrow \quad \beta l \approx z^2\left(\frac{l}{2a(l-a)}\right)$$

$$z^2 = 2\beta a(l-a)$$

Combine this with:

$$z = \frac{Fa(l-a)}{Hl}$$

Results in:

$$H = \frac{1}{\sqrt{2\beta a(l-a)}} \frac{Fa(l-a)}{l}$$

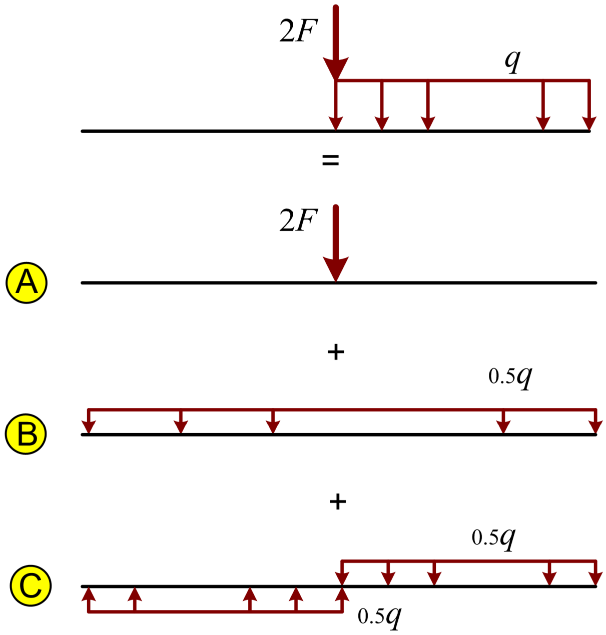
e) The value of z will be underestimated which will result in a higher value of H. So in the graph the upper line represents the approximate solution.

Problem 4

a) see the lecture material.

b) The wave length is 8 m. See the lecture material for explanation.

c) The settlement at the origin due to *F* and *q* can be found by superposition and by splitting the distributed load in a symmetrical and contra symmetrical part.



From this results a contribution to the settlement at the origin due to *q* of:

$\frac{1}{2} q / k$ (load case B)

Since the principle of superposition holds for these models we only have to add the displacement contribution due to *F*. This load case is a symmetrical load case so only half the model is solved. Use the alternative description for the homogeneous solution of the displacement field *w*: (using the other expression is fine as well)

$$w = Ae^{-\beta x} \sin\left(\beta x + \omega\right)$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\sqrt{2}\,\beta A e^{-\beta x} \sin\left(\beta x + \omega - \tfrac{1}{4}\pi\right)$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 2\beta^2 A e^{-\beta x} \sin\left(\beta x + \omega - \tfrac{1}{2}\pi\right) \qquad M = -EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

$$\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = -2\sqrt{2}\beta^3 A e^{-\beta x} \sin\left(\beta x + \omega - \tfrac{3}{4}\pi\right) \qquad V = -EI \frac{\mathrm{d}^3 w}{\mathrm{d}x^2}$$

To find the displacement field only two boundary conditions are needed at the origin. These conditions follow from the symmetry conditions. This results in:

$$\varphi(0) = 0 \quad \Rightarrow \quad \omega = \tfrac{1}{4}\pi$$

$$V(0) = -F \quad \Rightarrow \quad A = \frac{F}{2\sqrt{2} EI \beta^3}$$

With the constants solved the displacement at the origin due to F yields:

$$w(0) = \frac{F\,\tfrac{1}{2}\sqrt{2}}{2\sqrt{2}EI\beta^3} = \frac{F}{4EI\beta^3}$$

The total displacement w at the origin becomes:

$$w(0) = \frac{F}{4EI\beta^3} + \frac{q}{2k} = \frac{\beta F}{k} + \frac{q}{2k} = 0.01065 \text{ m}$$

d) The moment at the origin only results from the concentrated load since the symmetrical distributed load results in zero bending in the entire beam and the contra-symmetrical distributed load results in zero moment at the origin. The moment distribution due to F thus becomes:

$$M(x) = -\frac{F}{\sqrt{2}\beta} e^{-\beta x} \sin\left(\beta x - \tfrac{1}{4}\pi\right) \quad \Rightarrow \quad M(0) = \frac{F}{2\beta} = 63.66 \text{ kNm}$$

e) At $x = 4.0$ m the moment distribution due to the contra symmetrical distributed load is zero. This can be explained from the wave length of the beam (8 m) and a zero moment at the origin. So only the moment due to F has to be computed:

$$M(x) = -\frac{F}{\sqrt{2}\beta} e^{-\beta x} \sin\left(\beta x - \tfrac{1}{4}\pi\right) \quad \text{with: } \beta = \frac{\pi}{4}$$

$$M(4.0) = -\frac{F}{\sqrt{2}\beta} e^{-\pi} \sin\left(\tfrac{3}{4}\pi\right) = -\frac{F}{2\beta} e^{-\pi} = -2.75 \text{ kNm}$$