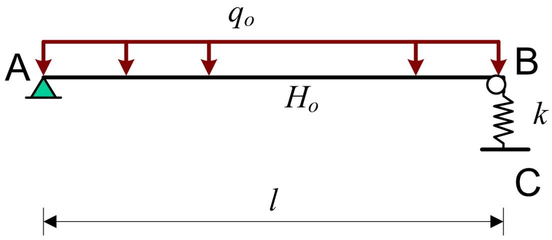
Exercises
DEC 2022

Problem 1: Force distribution (40 min)

The figure below visualizes a taut cable, connected to a fixed hinge at the left at A and through a roller (pulley) at B connected to a spring with a spring stiffness k. The spring remains vertical and is fixed to the ground at C. The cable is installed with a force H_o and any elongation of the cable is neglected. After installation, a distributed load is applied with magnitude q_o.



Given: H_o = 800 kN; q_o = 16 kN/m; k = 10000 kN/m; l = 8,0 m

Questions :

a) Calculate the elongation of the spring when the cable is installed.

After the installation phase the cable is loaded with the distributed load q_o. The additional elongation of the spring is measured and expressed as u. The length L of the loaded cable can be found with the approximation formula

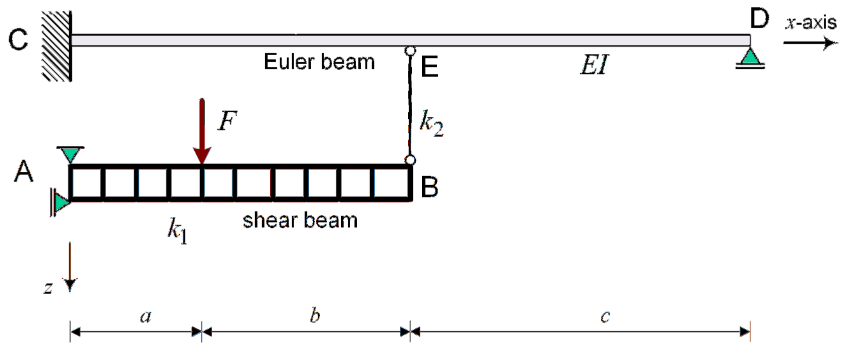
$$L = \int_{x=0}^{x=l} 1 + \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

In which w is the sagging of the cable with respect to the initial straight cable.

b) Model this problem to a differential equation.
c) Determine the boundary conditions.
d) Determine, by using MAPLE, the expression for $w(x)$ of the general solution and give the sagging of the cable at half of the span.
e) Which possibilities do you have to quickly check your solution. Verify this and show that the solution of d) is reasonable.

Problem 2: Compound systems (40 min)

The structure below presents a left clamped Euler-Bernoulli beam CD with bending stiffness EI. At E, the beam is connected to a two-force member which is connected to a shear beam at B. The shear beam, with shear stiffness k_1 is loaded with a concentrated force F at a distance a from the left support A of the shear beam.



Given:

$a = \frac{1}{4}l; \quad b = \frac{1}{4}l; \quad c = \frac{1}{2}l;$

$EI = 15000 \text{ kNm}^2; k_1 = 2000 \text{ kN/m}; k_2 = \infty; F = 40 \text{ kN}; l = 10 \text{ m}$

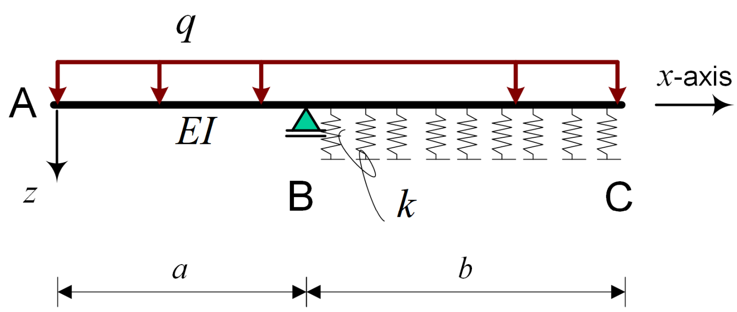
The two-force member has an axial stiffness k_2. Determine the vertical displacement $w(x)$, of the shear beam by using the differential equations. The questions below will guide you in your solution process.

Questions :

a) Explain shortly how the load bearing can be modelled of this structure. Indicate which equations are used.
b) What is the general solution of the displacement field $w(x)$ of the complete structure?
c) Identify the unknowns to be solved and formulate all conditions to solve the unknowns.
d) Solve the unknowns by using MAPLE and draw the displacement line of the structure. The line must be fulfilled the conditions.
e) Determine the final displacement of point B and the displacement of the shear beam at the point load.
f) Does the sagging fulfil the regular displacement condition?
g) What changes in your model if the rod is not rigid but has stiffness k_2?

Problem 3: Elastic foundation (40 min)

In the figure below a partial continuously elastic supported floor is presented which is supported at point B on a horizontal roller. We examine a strip of 1,0m width and model this as a partial continuously elastic supported Euler-Bernoulli beam with a bending stiffness of EI. At first, part BC may be assumed as an infinite length. The elastic support has a spring stiffness k. On the floor acts a distributed load q.



Given : $a = 10$ m; $b = 30$ m; $EI = 15000$ kNm^2; $k = 4000$ kN/m; $q = 8$ kN/m

Questions :

a) Where do you to expect the largest shear force in this beam?
b) Give the general solution for the function of the displacement for this problem. Indicate which integration constants appear.
c) Determine the required boundary / matching conditions and solve them using MAPLE.
d) Draw the M- and V-line.
e) Sketch the displacement line and determine the sagging of point A in mm.
f) Determine the magnitude of the shear force directly right of B as a function of the spring stiffness k and sketch this function. Put the spring-stiffness on the horizontal axis and the shear force on the vertical axis.
g) Investigate what changes in case the support at B is removed.

Answers

Problem 1

a) The extension of the spring is $u_o = 0.08$ m.

b) The force in the spring when load is applied becomes:

$$F_{veer} = H_o + k \times u \qquad \text{(} u \text{ is the additional extension of the spring due to loading)}$$

The horizontal component of the force in the cable can be found with:

$$-H \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = q \quad \text{with:}$$

$$T(l) = F_{spring}; \ \text{(due to the turning block or pulley)}$$

$$H = \sqrt{F_{spring}^2 - V(l)^2}\ ; \quad V(l) = \tfrac{1}{2} q \times l; \ \text{(based on equilibrium)}$$

c) The boundary conditions are:

$$x = 0; \quad w = 0;$$
$$x = l; \quad w = 0;$$

d) The general solution contains a particular and homogeneous solution as:

$$w = -\frac{8x^2}{\sqrt{100000000u^2 \ + \ 16000000u \ + \ 635904}} + C_1 + C_2$$

$$C_1 = \frac{4}{\sqrt{2484 + 62500u + 390625u^2}}$$

$$C_2 = 0$$

The one unknown, the additional extension u of the spring can be found based on the cable length which does not change. The leads to the following condition for $l + u$:

$$l + u = \int_{x=0}^{x=l} \sqrt{1 + w'^2}\,\mathrm{d}x$$

Solving this results in: $u = 0.00721820$ m

And subsequently: $w = -0.00919719x^2 + 0.07357753x$

Ad mid-span the cable shows a sag of 0,147 m due to 7 mm additional extension of the spring! The horizontal component of the force in the cable increases to 869.8 kN.

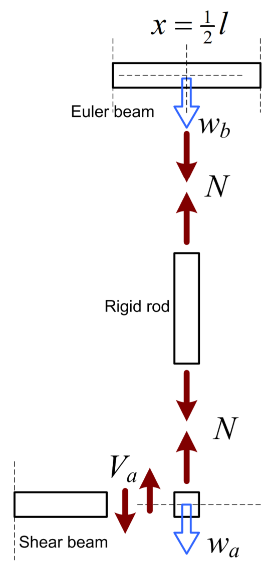
e) The well-known solution for an initial straight cable can be used as check:

$$w = \frac{\frac{1}{8} q l^2}{H_o} = 0{,}16 \text{ m}$$

Problem 2

a) The displacement of the beam in bending and the shear beam is only identical at the location of the link. The beams are therefore not continuous in parallel. So, this is a combination of basic systems linked at discrete locations. The beams are linked with a two-force member with an unknown force *N*. We can assume this force as the static redundant and solve the problem with the two basic problems linked by this force.

- Shear beam model with 2nd order ODE
- Beam in bending with 4th order ODE (or *forget-me-not's*)



b) Bending and shear:

$$EI\frac{\mathrm{d}^4 w_b}{\mathrm{d}x^4} = N \times Dirac\left(x - \tfrac{1}{2}l\right)$$

$$-k\frac{\mathrm{d}^2 w_a}{\mathrm{d}x^2} = F \times Dirac\left(x - \tfrac{1}{4}l\right)$$

Splitting the beams in fields, is of course also possible but with MAPLE the solution, including a Dirac is most efficient since we are interested in numerical results.

c) With this approach we have to solve 6 unknown integration constants and the unknown normal force *N* in the rigid rod (static redundant). In total seven unknowns. The required boundary conditions are:

Bending: $x = 0: \quad w_b = 0; \quad \varphi = 0; \quad x = l: \quad w_b = 0; \quad M = 0;$

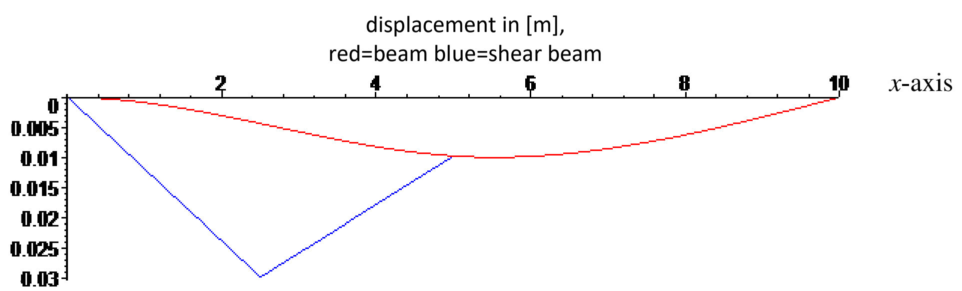
Shear: $x = 0: \quad w_a = 0; \quad x = \tfrac{1}{2}l: \quad V_a = -N$

The additional requirement can be found by the (rigid) coupling with the two-force member (see free body diagram to the right):

$$x = \tfrac{1}{2}l: \quad \begin{cases} k_2 = \infty & w_a = w_b \\ k_2 & N = k_2\left(w_a - w_b\right) \end{cases} \quad = \text{answer to question g)}$$

These equations can be solved with MAPLE.

d) Solving for a rigid rod results in the deflection line shown in the figure below.



e) The deflection at the location of the concentrated force is 0,03 m and at B 0,01 m.

f) With a span of 10 m the ratios are respectively 1/335 and 1/1023 which certainly satisfies a reasonable design requirement of 1/250 or 1/300.

Problem 3

For the cantilever part use a Euler-beam. For the elastic supported beam use the Winkler formulation. At the support use interface conditions to solve both displacements fields. In this answer only the basic steps to find the solution for the elastic supported part is shown.

To solve only the Winkler part a fast method can be used. At the support the displacement w will be zero and the moment M in the Winkler beam is $-\frac{1}{2}qa^2$. In which a is the length of the cantilever part. The compact notation for the general solution can be used:

$$w(x) = Ae^{-\beta x}\sin\left(\beta x+\omega\right)+\frac{q}{k} \qquad \text{with:} \quad 4\beta^4 = \frac{k}{EI}$$
$$M(x) = -2\beta^2 EIAe^{-\beta x}\sin\left(\beta x+\omega-\tfrac{\pi}{2}\right)$$

Apply the boundary conditions:

1) $w(0) = 0$
2) $M(0) = -\frac{1}{2}qa^2$

This results in:

1) $A\sin\omega = -q/k$
2) $A\sin\left(\omega-\frac{\pi}{2}\right) = \dfrac{qa^2}{4\beta^2 EI} \quad \Leftrightarrow \quad -A\cos\omega = \dfrac{qa^2}{4\beta^2 EI}$

From this follows:

$$\tan\omega = \frac{\sin\omega}{\cos\omega} = \frac{4\beta^2 EI}{ka^2} = \frac{1}{\left(\beta a\right)^2} \quad \Leftrightarrow \quad \omega = \arctan\left(\frac{1}{\left(\beta a\right)^2}\right)$$

From basic gonio follows (sketch a standard triangle …) :

$$\sin\omega = \frac{1}{\sqrt{\left(\beta a\right)^4+1}}$$

From 1) follows:

$$A = \frac{-q}{k\sin\omega} = -\frac{q\sqrt{\left(\beta a\right)^4+1}}{k}$$

Both constants are now solved. Diagrams can be drawn.