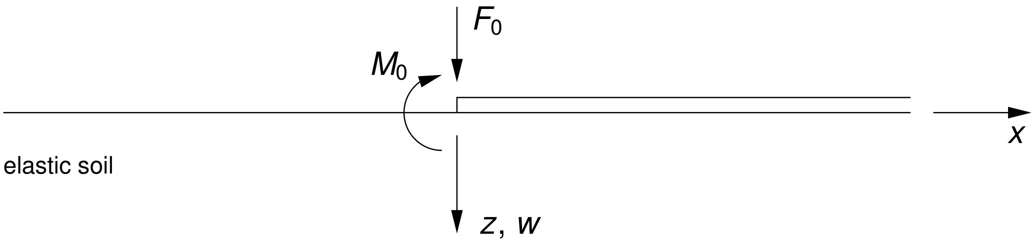
## General instructions

All pages of your solution must contain your name and your student number. Number each page of your solution and indicate the total number of pages used. Please write clearly and in English. Points for each question are in the left margin brackets.

## Questions

1. Beam on elastic foundation

[2] (a) Consider the semi-infinite beam on elastic foundation under the action of two concentrated loads, force and bending moment, depicted in the figure below.




Determine the expressions of the deflection $w(x)$, rotation $\theta = \mathrm{d}w(x)/\mathrm{d}x$, bending moment $M(x)$, and transverse shear force $V(x)$. Make use of the following symbols:

$$A_{\beta x} = \mathrm{e}^{-\beta x}(\cos\beta x + \sin\beta x), \qquad B_{\beta x} = \mathrm{e}^{-\beta x}\sin\beta x,$$
$$C_{\beta x} = \mathrm{e}^{-\beta x}(\cos\beta x - \sin\beta x), \qquad D_{\beta x} = \mathrm{e}^{-\beta x}\cos\beta x.$$

Also, note that

$$\frac{\mathrm{d}A_{\beta x}}{\mathrm{d}x} = -2\beta B_{\beta x}, \qquad \frac{\mathrm{d}B_{\beta x}}{\mathrm{d}x} = \beta C_{\beta x}, \qquad \frac{\mathrm{d}C_{\beta x}}{\mathrm{d}x} = -2\beta D_{\beta x}, \qquad \frac{\mathrm{d}D_{\beta x}}{\mathrm{d}x} = -\beta A_{\beta x}.$$

In the above relations, $\beta = \sqrt[4]{\frac{k}{4EI}}$ with $k$ the soil stiffness and $EI$ the flexural rigidity of the beam.

If you have done no mistakes, you should find that $V(x) = -F_0 C_{\beta x} - 2M_0\beta B_{\beta x}$.

**Solution:** The general solution for the deflection of an Euler-Bernoulli beam on elastic foundation can be written as

$$w(x) = \mathrm{e}^{\beta x}(C_1\cos\beta x + C_2\sin\beta x) + \mathrm{e}^{-\beta x}(C_3\cos\beta x + C_4\sin\beta x).$$

$C_1$, $C_2$, $C_3$ and $C_4$ are constants of integration which are determined by the boundary conditions.

Since $w(x) \to 0$ for $x \to \infty$, we must have $C_1 = C_2 = 0$. The boundary conditions at $x = 0$ determine $C_3$ and $C_4$:

$$M(0) = -EIw''(0) = M_0 \to C_4 = \frac{2\beta^2 M_0}{k},$$

$$V(0) = -EIw'''(0) = -F_0 \to C_3 = \frac{2\beta F_0}{k} - \frac{2\beta^2 M_0}{k}.$$

Armed with these expressions we find

$$w(x) = \frac{2\beta F_0}{k} D_{\beta x} - \frac{2\beta^2 M_0}{k} C_{\beta x},$$

$$w'(x) = -\frac{2\beta^2 F_0}{k} A_{\beta x} + \frac{4\beta^3 M_0}{k} D_{\beta x},$$

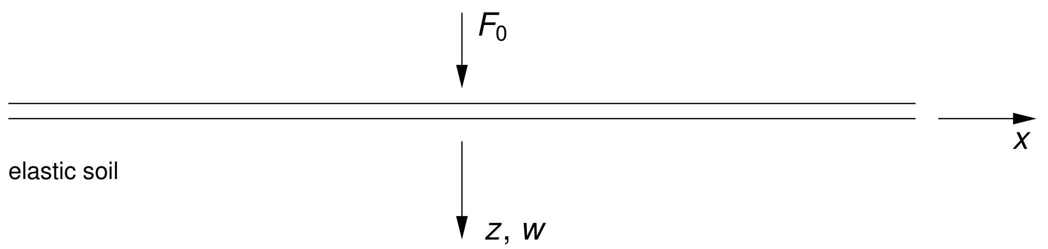
$$M(x) = -EIw''(x) = -\frac{F_0}{\beta} B_{\beta x} + M_0 A_{\beta x},$$

and

$$V(x) = -EIw'''(x) = -F_0 C_{\beta x} - 2M_0 \beta B_{\beta x},$$

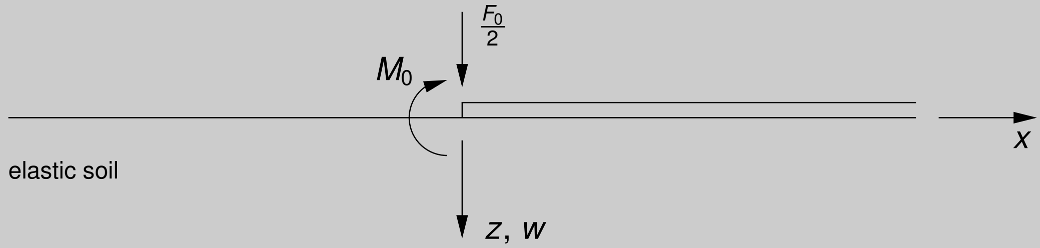
where a prime indicates the first derivative with respect to $x$, a double prime indicates the second derivative with respect to $x$ etc.

[2] (b) By using the solution from the previous question, determine the solution for the beam of infinite length on elastic foundation under the action of a concentrated force shown in the figure below.




**Solution:**

A beam of infinite length on elastic foundation under the action of a concentrated force can be equivalent to the beam of semi-infinite length on elastic foundation under the action of a concentrated force and a bending moment shown below.




The difference between these two cases is that in the former case the rotation or slope $w'(x)$ at the point of application of the force is zero. We can make use of this fact to derive a boundary condition at $x = 0$.
The expression for the slope derived in the previous question at $x = 0$ is

$$w'(0) = -\frac{2\beta^2 \frac{F_0}{2}}{k} + \frac{4\beta^3 M_0}{k}.$$

Note that we have used a load of intensity $F_0/2$.
By setting the slope to zero, we can derive the value of the bending moment that neutralise the slope created by the concentrated force. Proceeding along this line, we obtain

$$M_0 = \frac{F_0}{4\beta}.$$

Finally, using the expressions from the previous question with $F_0/2$ and $M_0 = F_0/4\beta$, we obtain the solution for the beam of infinite length:

$$w(x) = \frac{\beta F_0}{2k} A_{\beta x},$$

$$w'(x) = -\frac{\beta^2 F_0}{k} B_{\beta x},$$

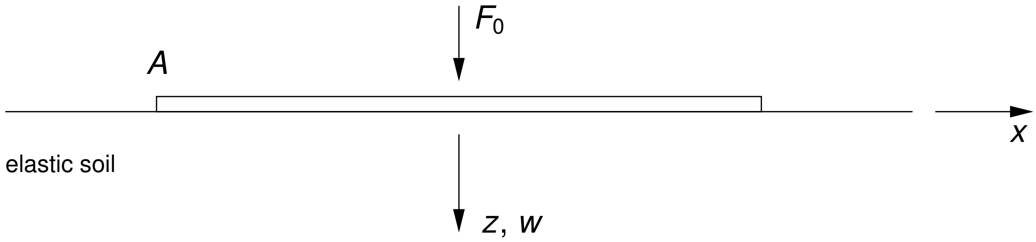
$$M(x) = -EIw''(x) = \frac{F_0}{4\beta} C_{\beta x},$$

$$V(x) = -EIw'''(x) = -\frac{F_0}{2} D_{\beta x}.$$

These expressions are valid for $x > 0$. The expressions for $x < 0$ are obtained from the symmetry and antisymmetry conditions: $w(x) = w(-x)$, $w'(x) = -w'(-x)$, $M(x) = M(-x)$, $V(x) = -V(-x)$.

[2] (c) Consider the beam of finite length $l$ on elastic foundation under the action of a concentrated load as depicted below.

What should the length of the beam be so that the deflection $w(x)$ derived for a beam of infinite length can be used with confidence in this case?

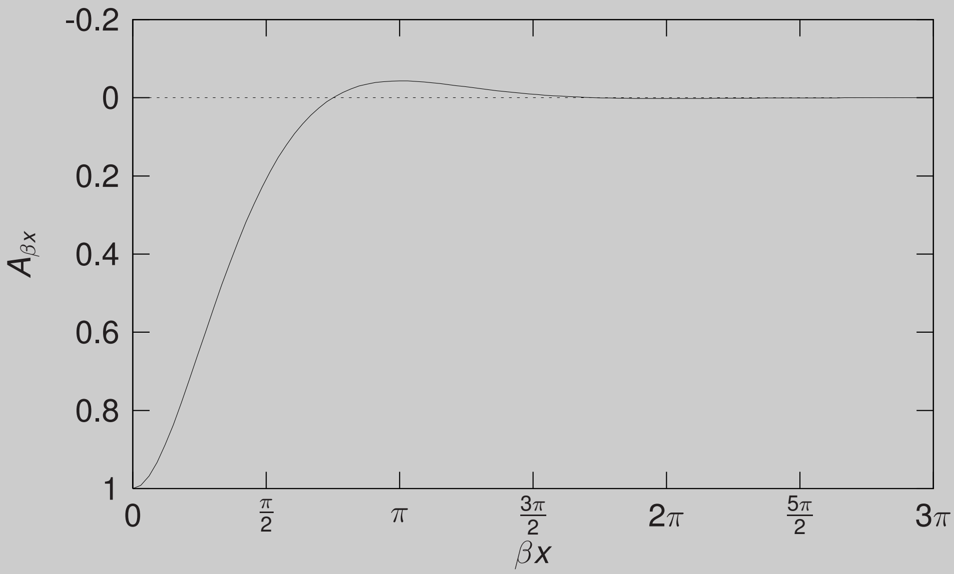



**Solution:**

The deflection $w(x)$ in an Euler-Bernoulli beam of infinite length on elastic foundation under the action of a concentrated load of intensity $F_0$ can be expressed by

$$w(x) = \frac{F_0\beta}{2k} e^{-\beta x} (\cos\beta x + \sin\beta x) = \frac{F_0\beta}{2k} A_{\beta x},$$

where $\beta = \sqrt[4]{\frac{k}{4EI}}$ with $k$ the soil stiffness and $EI$ the flexural rigidity of the beam. The function $A_{\beta x}$ is shown in the figure below.




To answer the question, we need to evaluate the function $A_{\beta x}$ at a few points.

| $\beta x$ | $A_{\beta x}$ | $\lvert A_{\beta x}\rvert$ [%] |
|---|---|---|
| 0 | $+0.10000E+01$ | 100.000 |
| $\frac{\pi}{2}$ | $+0.20788E+00$ | 20.788 |
| $\pi$ | $-0.43214E-01$ | 4.321 |
| $\frac{3\pi}{2}$ | $-0.89833E-02$ | 0.898 |
| $2\pi$ | $+0.18674E-02$ | 0.187 |
| $\frac{5\pi}{2}$ | $+0.38820E-03$ | 0.039 |
| $3\pi$ | $-0.80700E-04$ | 0.008 |

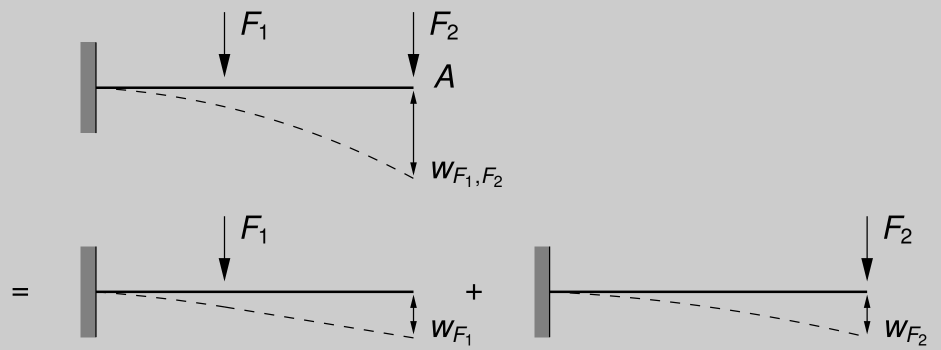
When $\beta x > \frac{3\pi}{2}$, $|A_{\beta x}| < 1\%$. This means that for points at a distance larger than $\frac{3}{2}\frac{\pi}{\beta}$ from the point of application of the concentrated force, the effect of the soil stiffness on the deflection can be neglected. Therefore, a beam of length $l > 2\frac{3}{2}\frac{\pi}{\beta}$ with a concentrated load applied at midspan exhibits approximately the same deflection curve as an infinitely long beam under the action of a concentrated load of the same intensity.

2. Principle of superposition

[1] (a) Give an example where the principle of superposition of displacements holds.

**Solution:**

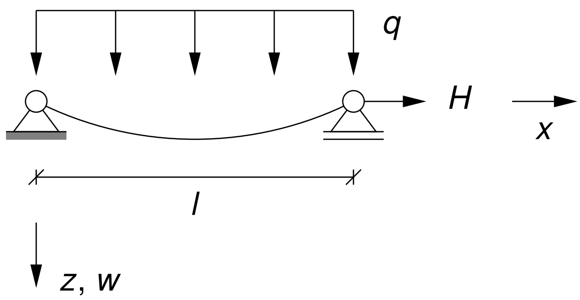
Amongst the many possibilities, see question 1(a) or consider the beam shown in the figure below. The material of the beam obeys Hooke's law, $E$ is the modulus of elasticity and $I$ is the moment of inertia. If a concentrated force $F$ is applied at the free end, at a distance $l$ from the clamped end, the deflection at the free end, $w = Fl^3/3EI$, is a linear function of $F$. Under these circumstances, the principle of superposition of deflections holds. Hence, if $w_j(i)$ indicates the transversal displacement at point $i$ caused by a force $j$, then the deflection at point $A$ caused by the forces $F_1$ and $F_2$ equals $w_{F1,F2}(A) = w_{F_1}(A) + w_{F_2}(A)$.




[3] (b) Consider the cable shown in the figure below. The left-hand side end is fixed

while the right-hand side can move horizontally and is the point of application of the horizontal force $H$. The cable is under the action of a distributed load $q$ which is expressed as a function of the cable tension $T$ at $x = l$ through $q = \lambda T(l)/l$, where $\lambda > 0$ is a load factor.

Find the expression of the cable deflection and the cable sag $f$. Does the principle of superposition of cable sags and horizontal components of the cable tension $T$ hold? Justify your answer.



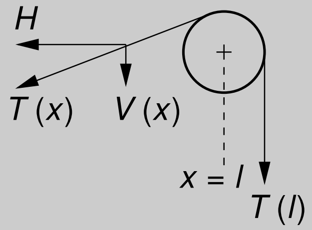

**Solution:** To answer this question we have to express the cable sag and the horizontal component of the cable tension $T$ as a function of the applied load. Given the expression of the deflection

$$w(x) = \frac{q}{2H} x (l - x),$$

the cable sag $f$, in this case, is equal to the deflection at midspan:

$$f = w(l/2) = \frac{ql^2}{8H} = \frac{\lambda T(l)\, l}{8H}.$$

By using Phytagoras' theorem we obtain the relation $H^2 = T^2(x) - V^2(x)$.




Given that the distributed load is expressed as a function of $T(l)$, we can express $V(x)$ at $x = l$ and factor the common term $T(l)$. Armed with the expression

$$V(x) = -qx + \frac{1}{2} ql$$

of the vertical component $V$ of the cable tension, we can determine its value at $x = l$ as

$$V(l) = -\frac{1}{2}ql = -\frac{1}{2}\lambda T(l).$$

By making use of the expression $H^2 = T^2(l) - V^2(l)$ and of the previous relation for $V(l)$, we are now ready to express the horizontal component of the cable tension as

$$H = T(l)\sqrt{1 - \frac{1}{4}\lambda^2}$$

from which the cable sag follows as

$$f = \frac{\lambda l}{8\sqrt{1 - \frac{1}{4}\lambda^2}}.$$

The principle of superposition is not valid since the relation between the cable sag $f$ or the horizontal component $H$ of the cable tension and the applied load, expressed through the load factor $\lambda$, is not a linear one.