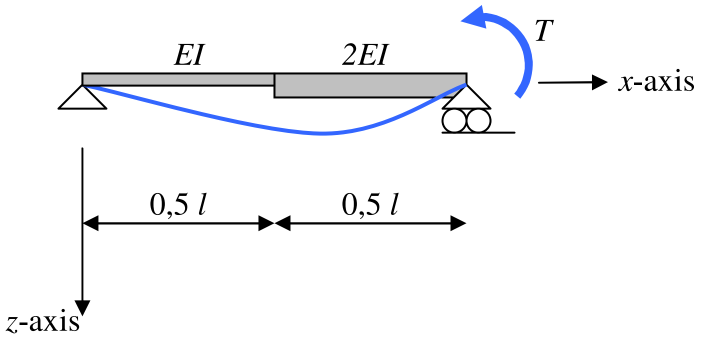
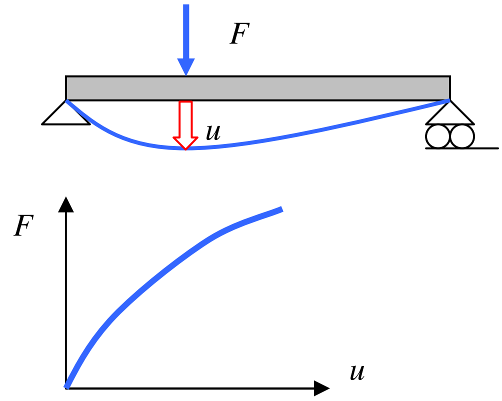
Work and energy method



Exercise 1 : Beam with a couple

Determine the rotation at the right support of the construction displayed on the right, caused by the couple T using Castigliano 's 2nd theorem.

Exercise 1 : Non-linear load-displacement diagram



A non-linear load-displacement diagram is given in the figure to the right. The relation between load and displacement is given as:

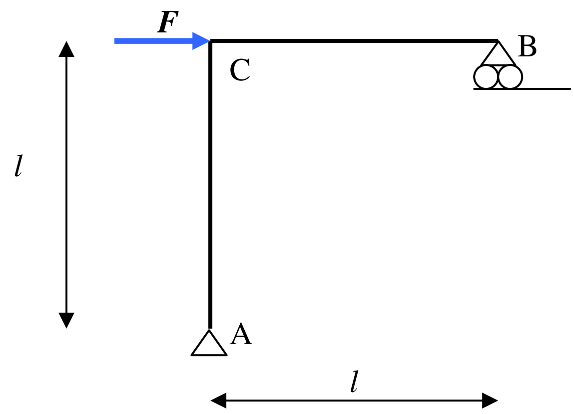
$$u = cF^2$$

Find:

a) The strain energy (in terms of u)
b) The complementary energy (in terms of F)

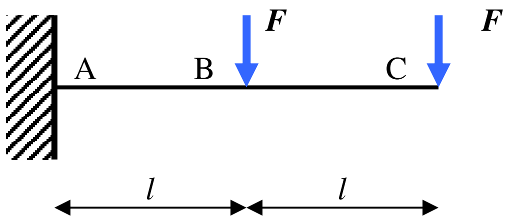
Exercise 2 : Horizontally loaded frame

Calculate the horizontal displacement at B using Castigliano's 2nd theorem. The bending stiffness of all beams is EI. Axial deformation (by the normal force) is neglected.



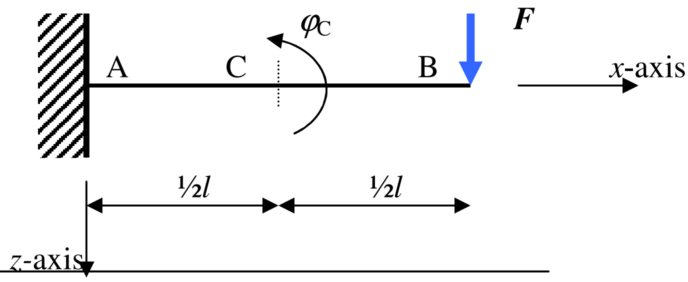
Exercise 3 : Cantilever beam 2

Calculate the displacement at B using the Castigliano's 2nd theorem. The bending stiffness of the prismatic beam is EI.



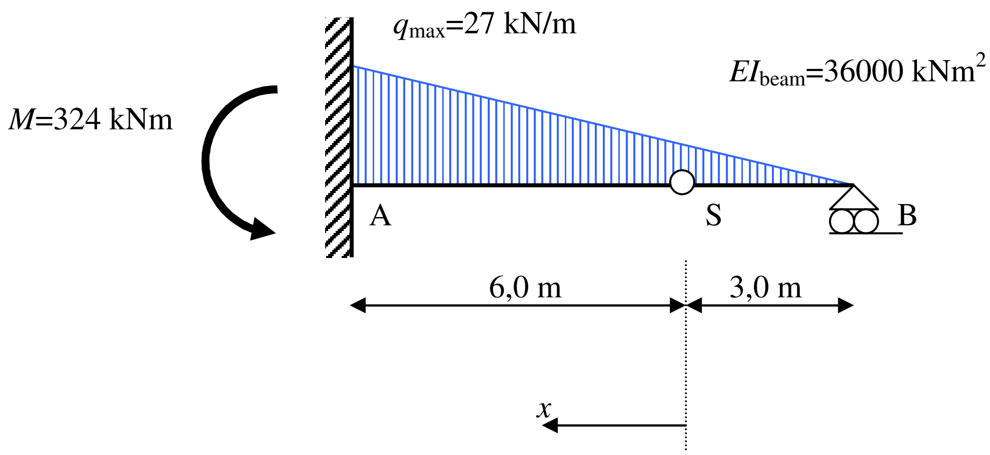
Exercise 4 : Cantilever beam 3

Calculate the rotation at C caused by a force at B. The bending stiffness of the prismatic beam is EI. Use Castigliano's 2nd theorem



Exercise 5 : Cantilever beam 4

In the figure a hinged beam with a distributed load *q(x)* is presented with a fully clamped support at A.



The moment distribution can be described as a function of *x [m]*:

$$M(x) = -9x - 4{,}5x^2 - 0{,}5x^3 \quad \text{[kNm]}$$

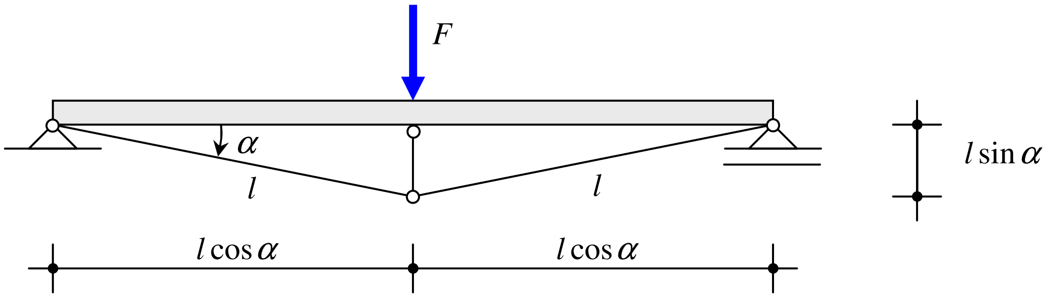
Hint : Have a good look at the coordinate system! A displacement starting at S to the left is positive!

Questions:

a) Draw the bending moment diagram for the beam.
b) Determine the deflection of the hinge S using the work method with a unit load.

Exercise 6: Strengthened beam

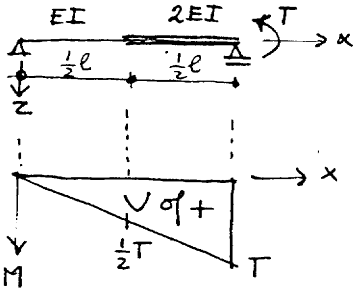
In the figure a strengthened beam with bending stiffness *EI* is presented with tension bars with length *l* of which the force distribution has to be determined. The axial stiffness of the beam is infinite and the axial stiffness of all bars is *EA*.



Questions:

a) Determine the static indeterminacy of this construction.
b) Describe how you will solve the compatibility conditions which are associated to the static redundant of the structure using Castigliano's theorem.
c) Solve the static redundant using the described method and give an expression using the parameters *F*, *EI*, *l*, *EA* and α.

Exercise 0 : Beam with a couple



$$M = T \cdot \frac{x}{\ell}$$

$$E_c = \int \frac{M^2}{2EI}\,dx$$

$$E_c = \int_0^{\frac{1}{2}\ell} \frac{\left(T\frac{x}{\ell}\right)^2}{2EI}\,dx + \int_{\frac{1}{2}\ell}^{\ell} \frac{\left(T\frac{x}{\ell}\right)^2}{4EI}\,dx$$

$$= \frac{1}{6}\frac{x^3T^2}{\ell^2EI}\Big|_0^{\frac{1}{2}\ell} + \frac{1}{12}\frac{x^3T^2}{\ell^2EI}\Big|_{\frac{1}{2}\ell}^{\ell}$$

$$= \frac{3}{32}\frac{T^2\ell}{EI}$$

Apply Castigliano:

$$E_C = \frac{3}{32}\frac{T^2\ell}{EI}$$

$$\varphi_C = \frac{dE_c}{dT} = \frac{3}{16}\frac{T\ell}{EI}$$

Exercise 1: Non-linear load-displacement diagram

$$u = cF^2 \qquad \text{of} \quad F = \sqrt{u/c} = \frac{u^{1/2}}{\sqrt{c}}$$

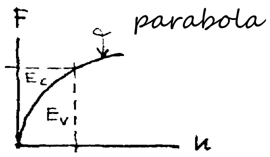
$$E_v = \int_0^u F\,du = \frac{1}{\sqrt{c}}\int_0^u u^{1/2}\,du = \frac{1}{\sqrt{c}}\left[\frac{1}{\frac{3}{2}}u^{\frac{3}{2}}\right]_0^u = \frac{2}{3}\sqrt{\frac{u^3}{c}}$$

$$E_c = \int_0^F u\,dF = \int_0^F cF^2\,dF = \frac{1}{3}cF^3\Big|_0^F = \frac{1}{3}cF^3$$

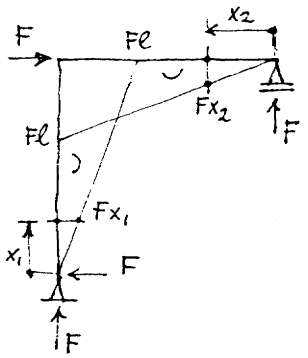
<u>Alternative (use area's):</u>

$$E_v = \frac{2}{3}Fu = \frac{2}{3}\sqrt{\frac{u}{c}}\cdot u = \frac{2}{3}\sqrt{\frac{u^3}{c}}$$

$$E_c = \frac{1}{3}Fu = \frac{1}{3}\cdot F\cdot cF^2 = \frac{1}{3}cF^3$$



Exercise 2 : Horizontally loaded frame



Find support reactions and moment distribution:

$$E_C = \int_{AB} \frac{M^2}{2EI}\,dx_1 + \int_{BC} \frac{M^2}{2EI}\,dx_2$$

$AB: \quad M = Fx_1 \qquad \left(\frac{\partial M}{\partial F} = x_1\right)$

$BC: \quad M = Fx_2 \qquad \left(\frac{\partial M}{\partial F} = x_2\right)$

The sign is not relevant

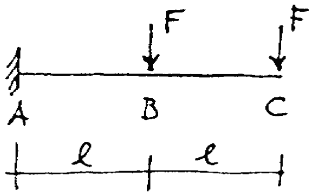
$$u_F = \frac{\partial E_c}{\partial F} = \int_{AB} \frac{M}{EI}\cdot\frac{\partial M}{\partial F}\,dx_1 + \int_{BC} \frac{M}{EI}\cdot\frac{\partial M}{\partial F}\,dx_2 \qquad *)$$

$$= \int_0^{\ell} \frac{Fx_1}{EI}\cdot x_1 \cdot dx + \int_0^{\ell} \frac{Fx_2}{EI}\cdot x_2\,dx_2$$

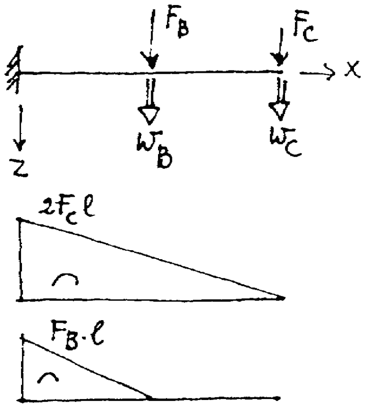
$$= \frac{F}{EI}\,\frac{1}{3}x_1^3\Big|_0^{\ell} + \frac{F}{EI}\cdot\frac{1}{3}x_2^3\Big|_0^{\ell} = \frac{2}{3}\frac{F\ell^3}{EI}$$

*) *The order of differentiation and integration can be changed to save time*

Exercise 3 : Cantilever beam 2



Find the deflection at B with Castigliano's second theorem. The beam is prismatic and has a bending stiffness EI.



Solution:

Give both forces a unique name:

$0 \leq x \leq \ell$

$$M = -F_B\,\ell\left(1-\frac{x}{\ell}\right) - 2F_C\,\ell\left(1-\frac{x}{2\ell}\right)$$

$\ell \leq x \leq 2\ell$

$$M = -2F_C\,\ell\left(1-\frac{x}{2\ell}\right)$$

$$E_C = \int_0^{\ell}\frac{M^2}{2EI}\,dx + \int_0^{2\ell}\frac{M^2}{2EI}\,dx$$

$$w_B = \frac{\partial E_c}{\partial F_b} = \int_0^{\ell}\frac{M}{EI}\underbrace{\frac{\partial M}{\partial F_B}}_{-\ell\left(1-\frac{x}{\ell}\right)}dx + \int_0^{2\ell}\frac{M}{EI}\underbrace{\frac{\partial M}{\partial F_B}}_{0}dx$$

$$w_B = \frac{1}{EI}\int_0^{\ell}\left[-F_B\,\ell\left(1-\frac{x}{\ell}\right) - 2F_C\,\ell\left(1-\frac{x}{2\ell}\right)\right]\left[-\ell\left(1-\frac{x}{\ell}\right)\right]dx$$

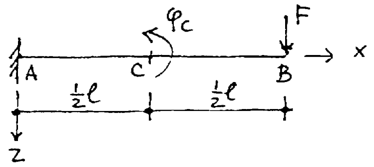
$$= \frac{F_B\ell^2}{EI}\int_0^{\ell}\left(1-2\frac{x}{\ell}+\frac{x^2}{\ell^2}\right)dx + \frac{2F_C\ell^2}{EI}\int_0^{\ell}\left(1-\frac{3}{2}\frac{x}{\ell}+\frac{1}{2}\frac{x^2}{\ell^2}\right)dx$$

$$= \frac{F_B\ell^2}{EI}\left(x-\frac{x^2}{\ell}+\frac{1}{3}\frac{x^3}{\ell^2}\right)\Bigg|_0^{\ell} + \frac{2F_C\ell^2}{EI}\left(x-\frac{3}{4}\frac{x^2}{\ell}+\frac{1}{6}\frac{x^3}{\ell^2}\right)\Bigg|_0^{\ell}$$

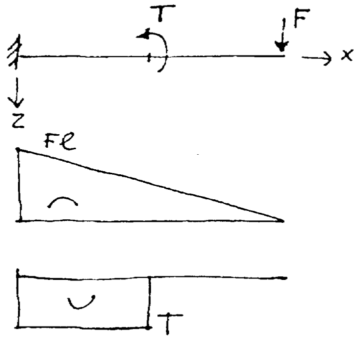
$$= \frac{1}{3}\frac{F_B\ell^3}{EI} + \frac{5}{6}\frac{F_C\ell^3}{EI} = \frac{7}{6}\frac{F\ell^3}{EI}$$

substitute the value $F_B = F_C = F$!

Exercise 4 : Cantilever beam 3



Find the rotation at C due to a load at B. The beam is prismatic and has a bending stiffness EI.



Solution:

Apply a couple T at C in the direction of the requested rotation.

$$\varphi_C = \frac{\partial E_c}{\partial T}$$

Substitute $T = 0$ in the answer obtained.

$$0 \leq x \leq \tfrac{1}{2}\ell \quad : \quad M = T - F(\ell - x) \qquad \left(\frac{\partial M}{\partial T} = 1\right)$$

$$x \geq \tfrac{1}{2}\ell \quad : \quad M = -F(\ell - x) \qquad \left(\frac{\partial M}{\partial T} = 0\right)$$

$$E_C = \int_0^{\frac{1}{2}\ell} \frac{M^2}{2EI}\,dx + \int_{\frac{1}{2}\ell}^{\ell} \frac{M^2}{2EI}\,dx$$

$$\varphi_C = \frac{\partial E_C}{\partial T} = \int_0^{\frac{1}{2}\ell} \frac{M}{EI} \underbrace{\frac{\partial M}{\partial T}}_{1} dx + \int_{\frac{1}{2}\ell}^{\ell} \frac{M}{EI} \underbrace{\frac{\partial M}{\partial T}}_{0} dx$$

$$= \int_0^{\frac{1}{2}\ell} \frac{T - F(\ell - x)}{EI} . 1 . dx = -\frac{3}{8}\frac{F\ell^2}{EI} + \frac{1}{2}\frac{T\ell}{EI}$$

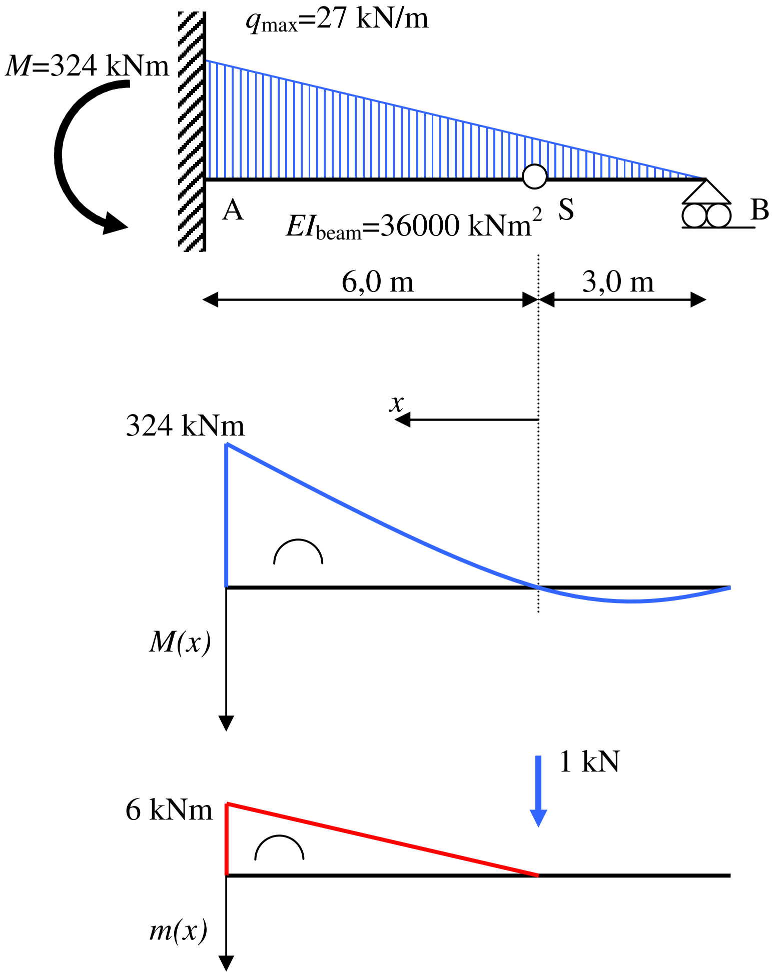
Substitute $T = 0$ to obtain the final answer:

$$\varphi_C = -\frac{3}{8}\frac{F\ell^2}{EI}$$

The couple T is only a dummy so therefore this method is also referred to as the "dummy load" method

Exercise 5 : Cantilever beam 4

a) The moment distribution is presented in blue in the graph to the right.



b) To find the displacement the work method with unit load is used:

$$w = \int_0^l \frac{m(x)M(x)}{EI}\,\mathrm{d}x$$

In this $M(x)$ is the moment distribution due to the actual loading, and $m(x)$ is the moment distribution due to a unit load applied at the hinge for which the deflection is asked for. This graph is presented in red.

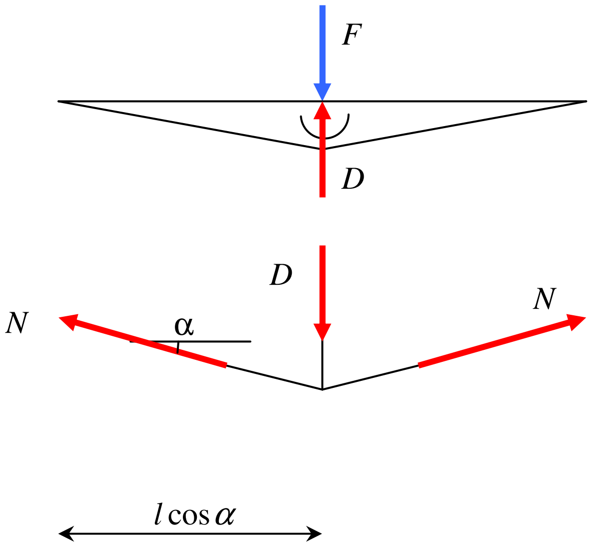
The table from the notes to find the solution for this integral can not be used so an analytical solution is required.

However part of the distribution $m(x)$ is zero so this reduces the complexity. Only part AS from 0 to 6 m has to be evaluated :

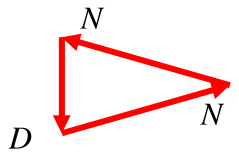
$$w = \frac{1}{EI}\int_0^6 (-x)\times(-9x-4{,}5x^2-0{,}5x^3)\,\mathrm{d}x = \frac{2883{,}6}{EI} = 80{,}1\ \mathrm{mm}$$

Exercise 6 : Strengthened beam

The beam is simply statically indeterminate. Assume the compressive force D in the vertical strut (two-force member) as the statically unknown. All forces in the truss system under the beam can be expressed in terms of this unknown D. With these forces the complementary strain energy can be expressed in terms of the unknown D. For the beam the moment distribution can be obtained including the concentrated load from the vertical strut.



$$N = \frac{D}{2\sin\alpha}$$



All bars have their own stiffness k from which the strain energy in the strengthened part can be obtained:

$$E_{v-1} = 2\cdot\frac{N^2}{2k_1} + \frac{D^2}{2k_2} \quad \text{with:} \quad k_1 = \frac{EA}{l}; \quad k_2 = \frac{EA}{l\sin\alpha}$$

The complementary energy stored in bending can be found with the moment distribution in the beam due to the load F and the statically unknown D:

$$E_{v-2} = \int_0^{2l\cos\alpha} \frac{M^2}{2EI}dx = 2\int_0^{l\cos\alpha} \frac{\left(\frac{1}{2}(F-D)\cdot x\right)^2}{2EI}dx = \frac{(F-D)^2\, l^3\cos^3\alpha}{12EI}$$

At mid span the deflection of the beam (in the direction of F) must be the same as that of the strengthening sub structure (in the direction of D) resulting in:

$$\frac{dE_{v-1}}{dD} = \frac{dE_{v-2}}{dF} \quad \Leftrightarrow$$

$$\frac{Dl}{2EA\sin^2\alpha} + \frac{Dl\sin\alpha}{EA} = \frac{(F-D)l^3\cos^3\alpha}{6EI}$$

$$D = \frac{EA\cdot l^2\cos^3\alpha\sin^2\alpha}{3EI + 6EI\sin^3\alpha + EA\cdot l^2\cos^3\alpha\sin^2\alpha}F$$