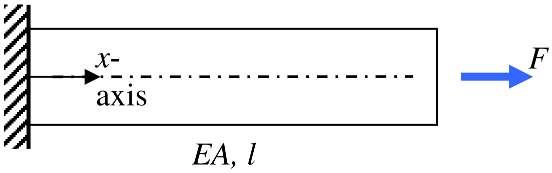
ENERGY PRINCIPLES

Exercise 1 : Bar loaded in tension



If the principle of minimum potential energy is applied to a limited number of kinematically admissible displacement fields equilibrium will not be met everywhere but on average it will be. This leads to an approximation of the displacements and force distribution in the structure.

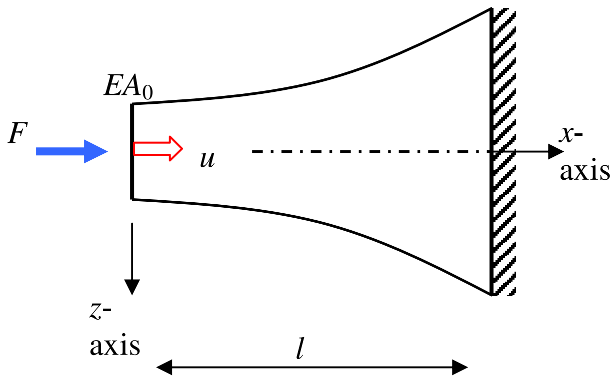
Calculate (approximate) the displacement distribution u and the normal force distribution N using the principle of minimum potential energy, assume the following two kinematically admissible displacement fields:

$$u(x) = a\frac{x^2}{l^2}$$

$$u(x) = a_1\frac{x}{l} + a_2\frac{x^2}{l^2}$$

exercise 2 : Non-prismatic cross-section

A non-prismatic bar as shown in the figure on the right is loaded in compression. The non-prismatic axial stiffness is given as:



$$EA(x) = \frac{2EA_o}{2-\dfrac{x}{l}}$$

The material is linear-elastic.

Calculate the displacement a using the principle of minimal potential energy, assuming the following kinematically admissible displacement field:

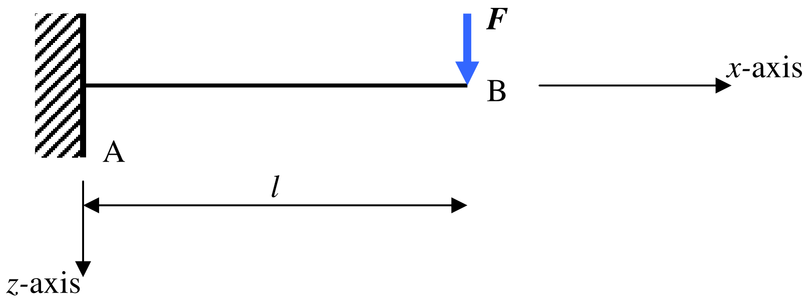
$$u(x) = a\left(1-\frac{x}{l}\right)$$

Use the following set of steps which together form the approach of the displacement method.

Kinematic equations
Constitutive equations
Equilibrium equations ↓ *DISPLACEMENT METHOD*

Exercise 3 : Cantilever beam

Use the principle of minimum potential energy to determine the displacement field w, the shear force $\boldsymbol{V}$ and the bending moment $\boldsymbol{M}$. Assume the following kinematically admissible displacement field:



$$w(x) = a\frac{x^2}{l^2}$$

Compare the result with the exact solution.

Exercise 4 : Cantilever beam (continued)

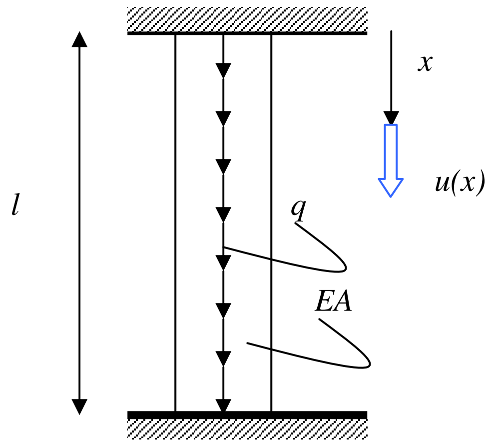
Using the beam of the previous exercise but now assume the following displacement field:

$$w(x) = a_o + a_1\frac{x}{l} + a_2\frac{x^2}{l^2} + a_3\frac{x^3}{l^3}$$

a) Is this displacement field kinematically admissible? If not modify it.
b) Use the principle of minimum potential energy to calculate the deflection function w.

Exercise 5 : Bar loaded in tension (2)

The displacements of the bar ends in the structure shown on the right are prohibited. On the prismatic column with axial stiffness EA a distributed load q acts over the entire length l. The normal force distribution in the column is calculated with the principle of minimum potential energy. For the displacements $u(x)$ in the x-direction we assume the following:



$$u(x) = \hat{u}\sin\frac{\pi x}{l} \qquad \text{amplitude } \hat{u}$$

The amplitude of this sinusoidal is a yet to be determined constant.

Answer the following questions:

a) Prove that the assumed displacement field satisfy to the kinematic boundary conditions.
b) Calculate the amplitude expressed in q, l and EA using the principle of minimum potential energy.
c) Draw the accompanying normal force diagram and calculate the characteristic values (with the correct signs!).

ANSWERS

Exercise 1 : Bar loaded in tension

The potential energy can be determined based on the assumed displacement field that satisfies the boundary conditions and is therefore a kinematically admissible displacement field. For the displacement field and strain field the following holds:

$$u = \frac{ax^2}{l^2}$$
$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2ax}{l^2}$$

The potential energy can now be written as:

$$V = \int_0^l \tfrac{1}{2} EA \times \varepsilon^2 dx - F \times a = \int_0^l \tfrac{1}{2} EA \times \left(\frac{2ax}{l^2} \right)^2 dx - F \times a \quad \Leftrightarrow$$
$$V = \frac{2EAa^2}{3l} - Fa$$

A stable equilibrium is only possible if a small disturbance from the state variable a doesn't cause a change in potential energy. This means from a mathematical point of view a stationary function:

$$\delta V = \frac{\mathrm{d}V}{\mathrm{d}a} \delta a = 0$$

This has to hold for every kinematically admissible variation of the state variable. This means that the variation of the potential energy can only be zero if the derivative of the potential energy to the state variable is zero:

$$\frac{\mathrm{d}V}{\mathrm{d}a} = 0$$

This is the principle of minimum potential energy. The result is:

$$\frac{\mathrm{d}V}{\mathrm{d}a} = \frac{4EAa}{3l} - F = 0 \quad \Leftrightarrow$$
$$a = \frac{3}{4} \frac{Fl}{EA}$$

The approximated displacement field is therefor:

$$u(x) = \frac{3}{4} \frac{FL}{EA} \frac{x^2}{l^2}$$

The potential energy thus has the following (negative) value:

$$V = \frac{-3}{8} \frac{F^2 l}{EA}$$

Exercise 2 : Non-prismatic cross-section

The potential energy can be determined based on the assumed displacement field that satisfies the boundary conditions and is therefore a kinematically admissible displacement field. Based on this displacement field the strain field becomes:

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-a}{l}$$

The potential energy can be written as:

$$V = \int_0^l \tfrac{1}{2} EA(x) \times \varepsilon^2 dx - F \times a$$

The axial stiffness is not a constant but a function of the location *x* and cannot be taken outside the integral. Solving for the total potential energy results in:

$$V = \int_0^l \tfrac{1}{2} \times \frac{2EA_o}{2 - \frac{x}{l}} \times \left(\frac{-a}{l} \right)^2 dx - Fa = \frac{-EA_o}{l} a^2 \int_0^l \frac{1}{2 - \frac{x}{l}} d\left(2 - \frac{x}{l} \right) - Fa$$

$$V = \frac{-EA_o}{l} a^2 \left[\ln(2 - \frac{x}{l} \right]_0^l - Fa = \frac{EA_o}{l} a^2 \ln 2 - Fa$$

A stable equilibrium is only possible if a small disturbance of the state variable *a* doesn't cause a change in potential energy. This means from a mathematical point of view a stationary function:

$$\delta V = \frac{\mathrm{d}V}{\mathrm{d}a} \delta a = 0$$

This has to hold for every kinematically admissible variation of the state variable. This means that the variation of the potential energy can only be zero if the derivative of the potential energy to the state variable is zero:

$$\frac{\mathrm{d}V}{\mathrm{d}a} = 0$$

This is the principle of minimal potential energy. The final result is:

$$\frac{\mathrm{d}V}{\mathrm{d}a} = \frac{2EA_o}{l} \ln 2 - F = 0 \quad \Leftrightarrow$$

$$a = \frac{1}{2 \ln 2} \frac{Fl}{EA_o}$$

Exercise 3 : Cantilever beam

The potential energy can be determined based on the assumed displacement field that satisfies the boundary conditions and is therefore a kinematically admissible displacement field. Based on this displacement field the curvature is:

$$\kappa = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{2a}{l^2}$$

The potential energy can be written as:

$$V = \int_0^l \tfrac{1}{2} EI \times \kappa^2 dx - F \times a = \int_0^l \tfrac{1}{2} EI \times \left(\frac{2a}{l^2} \right)^2 dx - F \times a$$

Solving for the total potential energy gives the following:

$$V = \frac{2EIa^2}{l^3} - Fa$$

A stable equilibrium is only possible if a small disturbance of the state variable *a* doesn't cause a change in the potential energy. This means from a mathematical point of view a stationary function:

$$\delta V = \frac{\mathrm{d}V}{\mathrm{d}a} \delta a = 0$$

This has to hold for every kinematically admissible variation of the state variable. This means that the variation of the potential energy can only be zero if the derivative of the potential energy to the state variable is zero:

$$\frac{\mathrm{d}V}{\mathrm{d}a} = 0$$

This is the principle of minimal potential energy. The result is:

$$\frac{\mathrm{d}V}{\mathrm{d}a} = \frac{4EIa}{l^3} - F = 0 \quad \Leftrightarrow$$

$$a = \frac{Fl^3}{4EI}$$

The potential energy thus has the following (negative) value:

$$V = \frac{-}{8} \frac{F^2 l^3}{EI}$$

Exercise 4 : Cantilever beam (continued)

A kinematically admissible displacement field has a *deflection* and *rotation* at a clamped end of zero. A admissible displacement field and the accompanying curvature could be:

$$w = a_2 \frac{x^2}{l^2} + a_3 \frac{x^3}{l^3}$$

$$\kappa = -\frac{\mathrm{d}^2 \mathrm{w}}{\mathrm{d}x^2} = -\frac{2a_2}{l^2} - \frac{6a_3 x}{l^3}$$

The potential energy can be written as:

$$V = \int_0^l \tfrac{1}{2} EI \times \kappa^2 dx - F \times (a_2 + a_3) =$$

$$V = \frac{6EIa_3{}^2}{l^3} + \frac{6EIa_2 a_3}{l^3} + \frac{2EIa_2^2}{l^3} - F \times \left(a_2 + a_3\right)$$

A stable equilibrium is only possible if a small disturbance of the state variables a_2 and a_3 doesn't cause a change in the potential energy. This means from a mathematical point of view a stationary function:

$$\delta V = \frac{\mathrm{d}V}{\mathrm{d}a_2}\delta a_2 + \frac{\mathrm{d}V}{\mathrm{d}a_3}\delta a_3 = 0$$

This has to hold for every kinematically admissible variation of the state variables. This means that the variation of the potential energy can only be zero if both the derivatives of the potential energy to the state variables are zero:

$$\frac{\mathrm{d}V}{\mathrm{d}a_2} = 0 \quad \text{en} \quad \frac{\mathrm{d}V}{\mathrm{d}a_3} = 0$$

This is the principle of minimal potential energy. The result is:

$$\frac{\mathrm{d}V}{\mathrm{d}a_2} = \frac{6EIa_3}{l^3} + \frac{4EIa_2}{l^3} - F = 0$$

$$\frac{\mathrm{d}V}{\mathrm{d}a_3} = \frac{12EIa_3}{l^3} + \frac{6EIa_2}{l^3} - F = 0$$

This results in a set of two equations with two unknowns. The unknowns a_2 and a_3 can be solved and the displacement field w(x) becomes:

$$a_2 = \frac{Fl^3}{2EI}; \quad a_3 = -\frac{Fl^3}{6EI}; \quad \Rightarrow \quad w(x) = \frac{Flx^2}{2EI} - \frac{Fx^3}{6EI}$$

The potential energy thus has the following (negative) value:

$$V = \frac{-}{6}\frac{F^2 l^3}{EI}$$

Exercise 5 : Bar loaded in tension (2)

The potential energy equation is:

$$V = \int_0^l \tfrac{1}{2} EA \times \varepsilon^2 dx - \int_0^l q \times u \, dx$$

$$V = \tfrac{1}{2} EA \frac{\pi^2}{l^2} \hat{u}^2 \int_0^l \cos^2 \frac{\pi x}{l} dx - q \times \hat{u} \int_0^l \sin \frac{\pi x}{l} dx$$

$$V = \tfrac{1}{4} EA \frac{\pi^2}{l} \hat{u}^2 - \frac{2ql}{\pi} \hat{u}$$

$$\delta V = \frac{dV}{d\hat{u}} \delta\hat{u} = \left(\tfrac{1}{2} EA \frac{\pi^2}{l} \hat{u} - \frac{2ql}{\pi} \right) \delta\hat{u} = 0$$

$$\hat{u} = \frac{4ql^2}{\pi^3 EA}$$

Whether the potential energy has reached a maximum or a minimum can be investigated by checking the second derivative:

$$\frac{d^2V}{d\hat{u}^2} = \tfrac{1}{2} EA \frac{\pi^2}{l^2} > 0$$

With this we have shown that the extreme value is indeed a minimum.