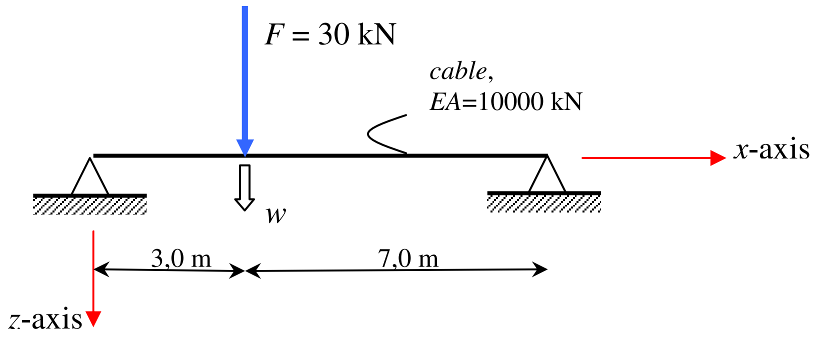
# EXERCISE 5: Work and Energy

A cable of exactly 10,5 m in length is placed between two fixed supports that are 10 meters apart. At 3 meters from the left support a point load F is applied as shown in the figure.

You are asked to use MAPLE to investigate the relation between the load $F$ and the displacement $w$ at the location of application of the point load.




*Questions :*

a) Determine the deflection at the point of application of the load as exactly as possible when we disregard the axial deformation of the cable. You can use any calculation model you like for this but it must be properly explained.

b) Due to the deflection the cable develops a stiffness for the load in the direction of the $z$-axis. Determine using the first theorem of Castigliano the stiffness of the cable for this load case at the point of application of the concentrated load for the given value of $F$.

c) Sketch the expected relation between $F$ and $w$ in a graph where the deflection is plotted on the horizontal axis for loads larger than the given $F$. What is in your opinion the qualitative influence of the axial stiffness of the cable on this relation.

## solution

a) The cable segments remain straight, there is discontinuity at the concentrated load. Neglecting axial deformation based upon equilibrium we find for the deflection at the point of application of the load:

$$z=\frac{M}{H}=\frac{F\times a\times(l-a)}{Hl}=\frac{F\times 3\times 7}{10H} \qquad (1)$$

This deflection can be found using the given cable length:

$$10,5=\sqrt{7^2+z^2}+\sqrt{3^2+z^2} \quad \Leftrightarrow z=1,48\text{ m} \qquad (2)$$

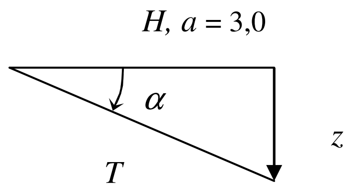
b) From (1) and (2) it follows that the horizontal component of the cable force *H* is:

$$H=42,5\text{ kN}$$

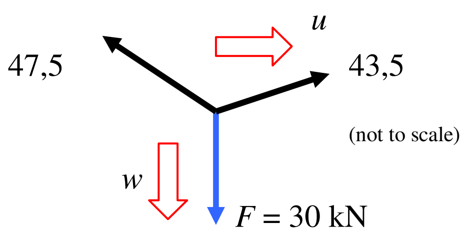
The tension in the cable segments left and right of the load can now be determined:

$$T_{left}=\frac{H}{\cos\alpha}=47,5\text{ kN} \quad with:\tan\alpha=\frac{z}{a}$$

$$T_{right}=\frac{H}{\cos\beta}=43,5\text{ kN} \quad with:\tan\beta=\frac{z}{l-a}$$




With this the following equilibrium situation for the deformed cable at the location of the load is found:




The cable segments at an angle can be considered as springs with a certain stiffness. These stiffnesses follow from the given axial stiffness *EA* and the length of the segments according to:

$$k_{left}=\frac{EA}{\sqrt{3^2+1,48^2}}; \quad k_{right}=\frac{EA}{\sqrt{7^2+1,48^2}}$$

An extra displacement *w* and *u* will give an elongation in the left and right segment of:

$$\Delta l_{left}=w\times\sin\alpha+u\times\cos\alpha; \quad \Delta l_{right}=w\times\sin\beta-u\times\cos\beta$$

Here the stiffness is found using Castigliano's first theorem. The strain energy stored in both segments, expressed in terms of deflections *u* and *w,* can be determined as follows:

$$E_v=\tfrac{1}{2}k_{left}\times\Delta l_{left}^2+\tfrac{1}{2}k_{right}\times\Delta l_{right}^2$$

Applying Castigliano theorem gives the stiffness relation between the external force on the system at $x=a$ and the horizontal and vertical displacement of the cable at that point:

$$
\begin{aligned}
F_x &= \frac{\partial E_v}{\partial u} = 3742,09u + 903,23w \\
F_z &= \frac{\partial E_v}{\partial w} = 903,23u + 644,99w
\end{aligned}
\quad \Leftrightarrow \quad
\begin{bmatrix} F_x \\ F_z \end{bmatrix} = \begin{bmatrix} k_{xx} & k_{xz} \\ k_{zx} & k_{zz} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}
$$

For this load case the horizontal force is equal to zero and the vertical force is equal to 30 kN. The following system of equations is found:

$$
\begin{bmatrix} 3742,09 & 903,23 \\ 903,23 & 644,99 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 30 \end{bmatrix}
$$

Solving this system results in the following displacements:

$$
u = -0,0169; \quad w = 0,0703
$$

At a structural level the relation between the force and the displacement can be regarded as a stiffness relation, $F = k \times w$ :

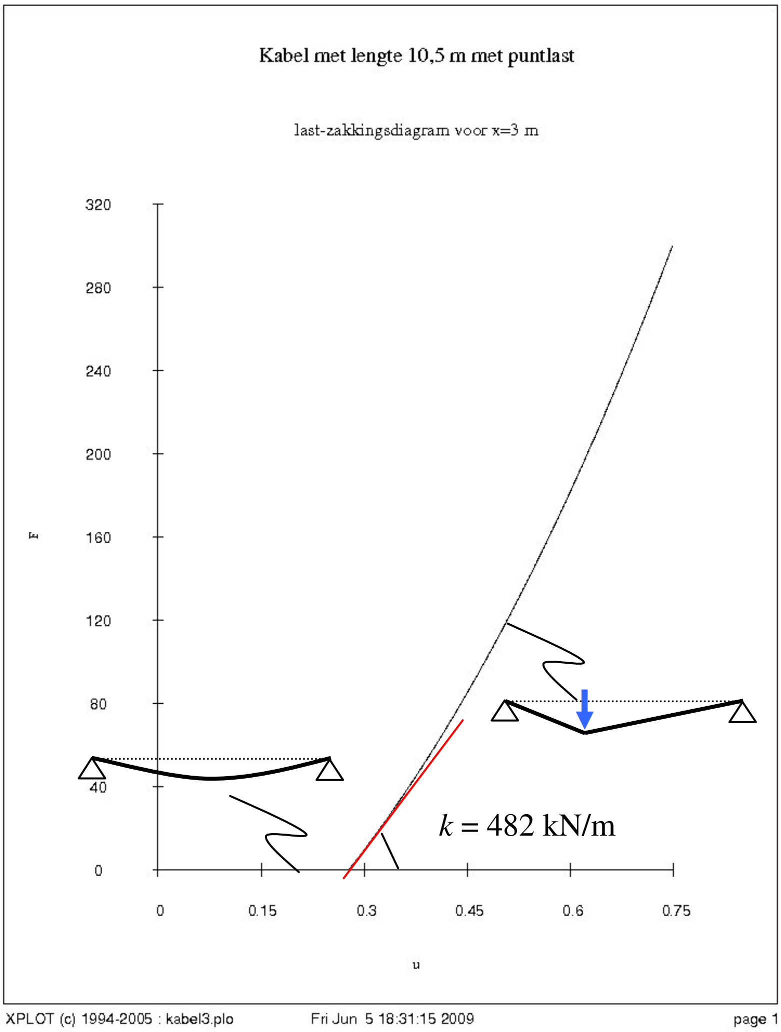
$$
k = \frac{F}{w} = \frac{30}{0,0703} = 426 \text{ kN/m}
$$

This stiffness can also be found by placing both angled springs as a truss structure in for example MatrixFrame. The stiffness is underestimated though because additional stiffness is created due to the geometric non linearity of the cable, which isn’t taken into account. The real stiffness will be higher.

c) The stiffness relation of the cable is *non-linear*. With increasing displacement the stiffness of the cable will increase. If the axial stiffness of the cable is taken into account the displacement will increase and the resulting stiffness will be lower.

> ***Note:***
> *From the question is not clear at what point the stiffness is requested. From the answer given it is clear that the point of application of the load is meant. On the next page the result is given for increasing load.*

*If the load increases a nonlinear computation is required*[1] *. To find a realistic solution first the parabolic shape of the unloaded cable of 10.5 m is computed and used.Next the load is increased up to 300 kN. This is 10 times the required load and thus will show the influence of the nonlinear behaviour.*




*From the graph it becomes clear that the unloaded structure has no stiffness. The shape of the cable first has to change from the parabolic shape in to the bi-linear shape which belongs to the concentrated loading. Starting from 0,27 m the system develops stiffness (with respect to the initial parabola).The slope of the diagram equals to the stiffness and a value of 482 kN/m is found which is slightly larger than the predicted value.*

[1] This calculation was executed with the nonlinear program FEMDEM, © J.W. Welleman