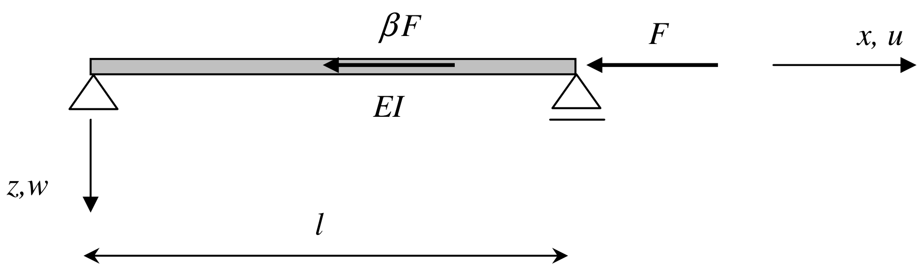
# Exercise 7: Work and Energy

In the figure below a beam loaded in bending and compression is shown. The beam is loaded at the right end and at mid span by two axial (compressive) loads as indicated. The normal force in the beam is therefore not constant. The axial deformation is neglected.




*Question*

Use a work / energy method to find an expression for the buckling load. Draw the relation between the factor $\beta$ and the buckling load $F$ in a graph.

## Solution

*Theory*
An equilibrium is stable when any transition to a close kinematic allowable state takes more strain energy than the work performed by the load.

When we apply this to a beam loaded in compression as in the figure below the concentrated load will perform work due to the axial displacement and the *buckling beam* will take up strain energy due to *bending* and *extension.*

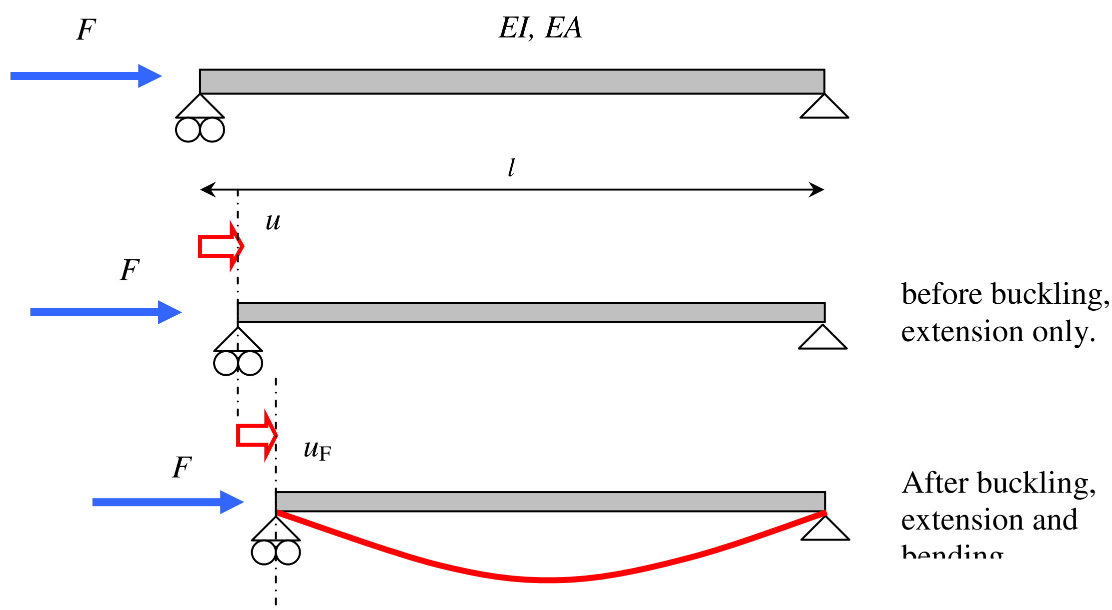



**Figure 1** : bending beam loaded in compression, Euler buckling beam.

Just before buckling there is only deformation due to extension, the beam is not yet bent. The expression for the strain energy can we written as:

$$E_v = \int_0^l \tfrac{1}{2} EA\varepsilon^2 dx$$

After buckling there is a curvature and the strain energy is equal to:

$$E_v = \int_0^l \tfrac{1}{2} EA\varepsilon^2 dx + \int_0^l \tfrac{1}{2} EI\kappa^2 dx$$

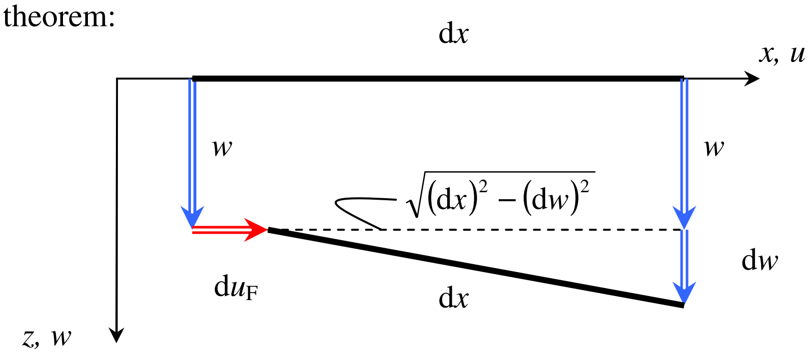
When the axial load is slowly increased the normal force in the beam just before and just after buckling will be the same. The axial strain is thus the same for both situations which means that the contribution to the strain energy just before and just after buckling is also the same. This means that in the transition of a straight beam to a bent beam there is an increase in the strain energy that is equal to only the strain energy due to bending:

$$\Delta E_v = \int_0^l \tfrac{1}{2} EI\kappa^2 \mathrm{d}x = \int_0^l \tfrac{1}{2} EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x$$

This strain energy is equal to the work performed by the concentrated load in the transition from the straight to the bent situation. This amount of work is equal to:

$$A = F \times u_{\mathrm{F}}$$

This horizontal displacement can be expressed in the vertical deflection using Pythagoras theorem:




$$\mathrm{d}u_{\mathrm{F}} = \mathrm{d}x - \sqrt{(\mathrm{d}x)^2 - (\mathrm{d}w)^2} = \\ = \left\{1 - \sqrt{1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2}\right\} \mathrm{d}x$$

**Figure 2** : small horizontal and vertical displacements.

The length of the beam is obviously the same before and after buckling.

The found expression for the horizontal displacement contains a complicated square root term with a quadratic derivative. This expression can be approximated using a Taylor-expansion (by neglecting the higher order terms) to:

$$\mathrm{d}u_{\mathrm{F}} = \left\{1 - \sqrt{1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2}\right\} \mathrm{d}x \cong \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

The total horizontal displacement after integrating over the length of the beam is equal to:

$$u_{\mathrm{F}} = \int_0^l \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

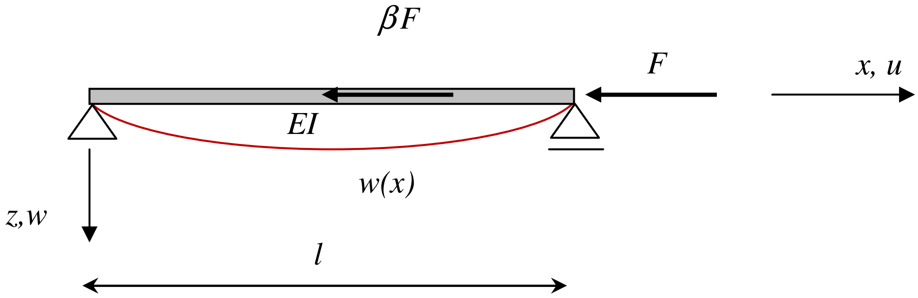
Setting the work done due to this horizontal displacement equal to the strain energy due to bending will give us the equation with which we can find the buckling load:

$$\Delta E_v = A \quad \Leftrightarrow \quad \int_0^l \tfrac{1}{2} EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = F \times \int_0^l \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

Equilibrium is just possible when the work performed is equal to the strain energy.

This approach is named after Rayleigh. By assuming a buckling field *w(x)* we can find a buckling load. This buckling load is higher than the real buckling load and that is something to take into account because this makes this approach inherently unsafe! The better the assumed displacement field fits the real buckling shape, the better the approximation of the buckling load will be. The buckling form must obviously adhere to the kinematic demands at the boundaries but will generally not adhere to all the equilibrium demands. This means small variations can occur. If the real buckling shape is assumed the real buckling load will be found.

*Application*
We apply the theory to our problem.




For this assignment we assume the following displacement field:

$$w(x) = C\sin\left(\frac{\pi x}{l}\right)$$

This will lead to the following axial deformations along the beam axis:

$$u(x) = -\int_0^x \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x = -\frac{C^2\pi}{4l^2}\left[l\cos(\frac{\pi x}{l})\sin(\frac{\pi x}{l}) + \pi x\right]$$

$$u(\tfrac{1}{2}l) = -\frac{C^2\pi^2}{8l}$$

$$u(l) = -\frac{C^2\pi^2}{4l}$$

Points on the beam axis will thus move left due to bending. Both concentrated loads that act in the negative $x$-direction, will thus perform positive work:

$$A_{uitw} = (-\beta F)\times\frac{-C^2\pi^2}{8l} + (-F)\times\frac{-C^2\pi^2}{4l}$$

$$A_{uitw} = \frac{C^2F\pi^2}{8l}(\beta+2) \qquad (1)$$

This amount of work must equal the stored strain energy due to bending:

$$\Delta E_v = \int_0^l \tfrac{1}{2}EI\kappa^2 \mathrm{d}x = \int_0^l \tfrac{1}{2}EI\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = \frac{C^2\pi^4 EI}{4l^3} \qquad (2)$$

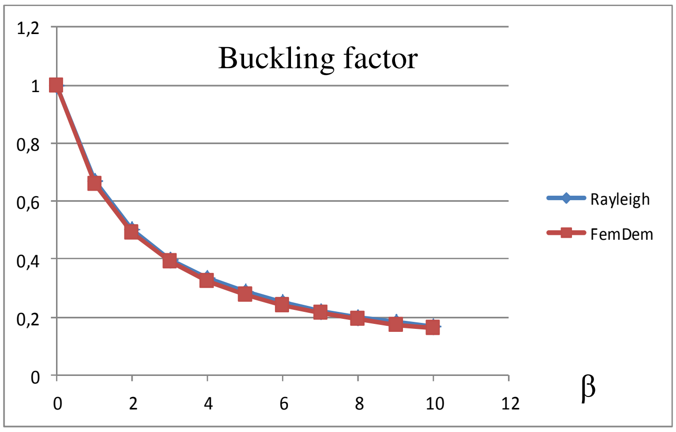
Setting (1) and (2) to be equal:

$$F_k = \frac{2\pi^2 EI}{(\beta+2)l^2} = \alpha\frac{\pi^2 EI}{l^2} \quad \text{with:} \quad \alpha = \frac{2}{\beta+2}$$

This result can be written in the form of a buckling factor α times the Euler buckling load with only the load $F$ on the right side of the beam.

This very simple result is tested against numerical results using a finite element program. In the following table the differences are shown and in a graph the difference between the Rayleigh buckling factor and the numerical buckling factor is shown for various values of β :

| β | Rayleigh | FemDem | difference % |
|---|---|---|---|
| 0 | 0,99978678 | 0,996240538 | 0,35 |
| 1 | 0,666693388 | 0,660614117 | 0,91 |
| 2 | 0,500020041 | 0,491914347 | 1,62 |
| 3 | 0,399965372 | 0,391353072 | 2,15 |
| 4 | 0,333346694 | 0,324734394 | 2,58 |
| 5 | 0,285725738 | 0,27736674 | 2,93 |
| 6 | 0,250010021 | 0,242157629 | 3,14 |
| 7 | 0,222146695 | 0,214547606 | 3,42 |
| 8 | 0,199856035 | 0,192763552 | 3,55 |
| 9 | 0,181871525 | 0,175032345 | 3,76 |
| 10 | 0,166673347 | 0,16008747 | 3,95 |




The solution is fairly accurate, the difference slowly increases to 3,5% for a point load halfway acorss the beam which is ten times larger than the point load on the right side of the beam. From the results it can be seen the numerical results are all lower than the result from Rayleigh's approach. This fits with our expectations from the theory.

*Real displacement field*
The assumed sinusoidal shape will not be the real displacement field because both parts of the beam have a different normal force. Both beam parts thus have a different differential equation:

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}+(\beta+1)F\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=0 \qquad 0\le x\le \tfrac{1}{2}l$$

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}+F\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=0 \qquad \tfrac{1}{2}l\le x\le l$$

The general solution of both beam segments thus results in solutions with sinus and cosinus terms with different periods.

$$w_1(x)=C_1+C_2x+C_3\sin(\alpha_1 x)+C_3\cos(\alpha_1 x) \quad \wedge \quad \alpha_1^2=\frac{(1+\beta)F}{EI} \qquad 0\le x\le \tfrac{1}{2}l$$

$$w_2(x)=D_1+D_2x+D_3\sin(\alpha_2 x)+D_3\cos(\alpha_2 x) \wedge \quad \alpha_2^2=\frac{F}{EI} \qquad \tfrac{1}{2}l\le x\le l$$

The assumed solution with only a sinus term with a single period can therefore never be correct. Further evaluation of this method is not asked for. The method should be known though, set up the boundary conditions and interface conditions that both fields must satisfy. A homogeneous system of equations results which can only have a non-trivial solution if the determinant of the system is zero. Solving the characteristic equation gives us an expression for the buckling load. This will not give a simple solution which is why the Rayleigh approach is more elegant.