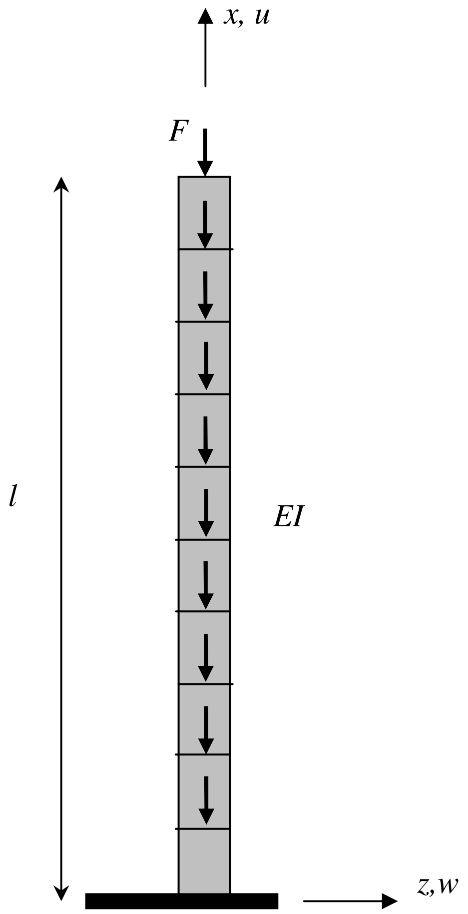
Exercise 8: work and energy

In the figure below we a schematic model of an elevator shaft in a building is shown. Every floor of the building transmits a load F onto this shaft. We assume that the elevator shaft is the component that takes care of the stability of the entire building. Any effects on the stability in the structure is added into the load F. The normal force in the shaft is not constant due to this load case and the general route to determine the buckling load using the differential equation for a constant normal force is not applicable.



Question

Determine an expression for the buckling load using a work and energy method and use this expression to find an expression for the buckling length of this column.

Solution

Theory
An equilibrium is stable when any transition to a close kinematic allowable state takes more strain energy than the performed work by the load.

When we apply this to a beam loaded in compression as in the figure below the concentrated load will perform work due to the axial displacement and the *buckling beam* will take up strain energy due to *bending* and *extension.*

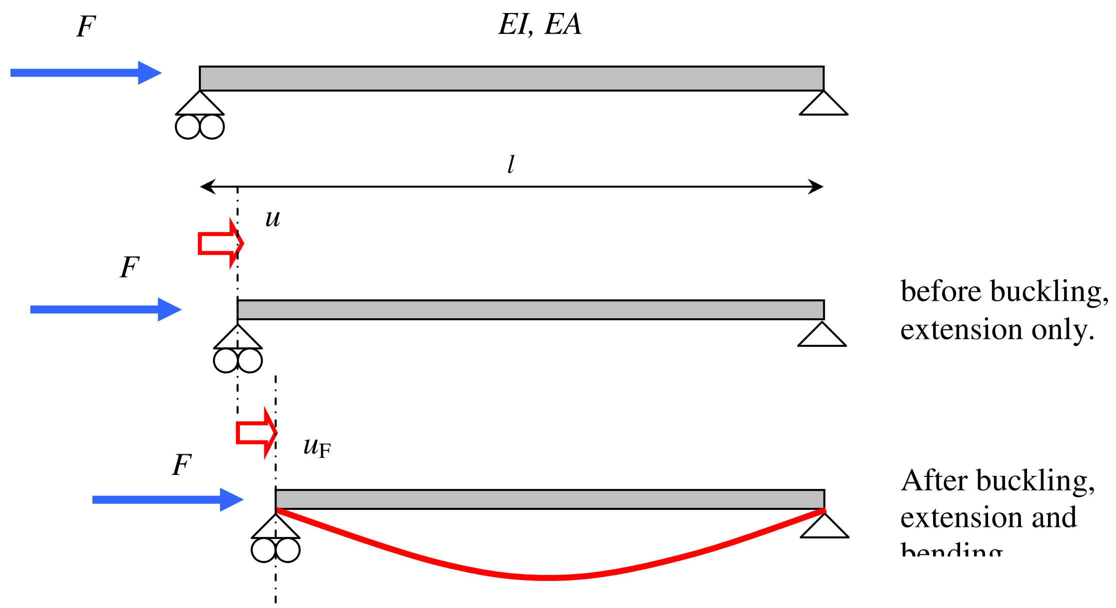


Figure 1 : bending beam loaded in compression, Euler buckling beam.

Just before buckling there is only deformation due to extension, the beam is not yet bent. For the strain energy the following holds:

$$E_v = \int_0^l \tfrac{1}{2} EA\varepsilon^2 dx$$

After buckling there is a curvature and the strain energy is then:

$$E_v = \int_0^l \tfrac{1}{2} EA\varepsilon^2 dx + \int_0^l \tfrac{1}{2} EI\kappa^2 dx$$

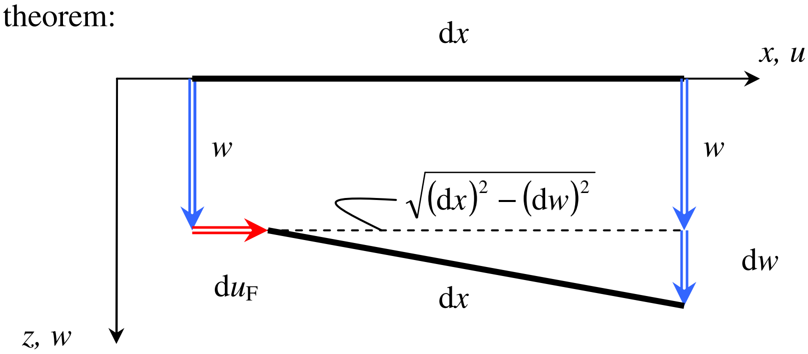
When the axial load is slowly increased the normal force in the beam just before and just after buckling will be the same. The axial strain is thus the same for both situations which means that the contribution of the strain energy just before and just after buckling is also the same. This means that in the transition of a straight beam to a bent beam there is an increase in the strain energy that is equal to only the strain energy due to bending:

$$\Delta E_v = \int_0^l \tfrac{1}{2} EI\kappa^2 \mathrm{d}x = \int_0^l \tfrac{1}{2} EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x$$

This strain energy is equal to the work performed by the concentrated load in the transition from the straight to the bent situation. This amount of work is equal to:

$$A = F \times u_{\mathrm{F}}$$

This horizontal displacement can be expressed in the vertical deflection using Pythagoras theorem:



$$\mathrm{d}u_{\mathrm{F}} = \mathrm{d}x - \sqrt{(\mathrm{d}x)^2 - (\mathrm{d}w)^2} = \\ = \left\{1 - \sqrt{1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2}\right\}\mathrm{d}x$$

Figure 2 : small horizontal and vertical displacements.

The length of the beam is obviously the same before and after buckling.

The found expression for the horizontal displacement contains a complicated square root term with a quadratic derivative. This expression can be approximated using a Taylor-expansion (by neglecting the higher order terms) to:

$$\mathrm{d}u_{\mathrm{F}} = \left\{1 - \sqrt{1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2}\right\}\mathrm{d}x \cong \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

The total horizontal displacement after integrating over the length of the beam is equal to:

$$u_{\mathrm{F}} = \int_0^l \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

Setting the work done due to this horizontal displacement equal to the strain energy due to bending will give us the equation with which we can find the buckling load:

$$\Delta E_v = A \quad \Leftrightarrow \quad \int_0^l \tfrac{1}{2} EI \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = F \times \int_0^l \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

Equilibrium is just possible when the work performed is equal to the strain energy.

This approach is named after Rayleigh. By assuming a buckling field *w(x)* we can find a buckling load. This buckling load is higher than the real buckling load and that is something to take into account because this makes this approach inherently unsafe! The better the assumed displacement field fits the real buckling shape, the better the approximation of the buckling load will be. The buckling form must obviously adhere to the kinematic demands at the boundaries but will generally not adhere to all the equilibrium demands. This means small variations can occur. If the real buckling shape is assumed the real buckling load will be found.

Application

The above theory is now applied to our elevator shaft. For simplicity sake we only draw two out of the ten applied forces.

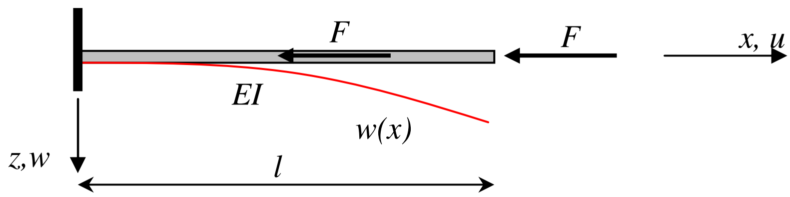


Figure 3 : buckling shape

For this exercise we assume a cosine shaped buckling shape that satisfies the kinematic boundary conditions:

$$w(x) = C\left[1-\cos\left(\frac{\pi x}{2l}\right)\right]$$

This shape will lead to axial displacements along the beam-axis:

$$u(x) = -\int_0^x \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x = -\frac{C^2\pi}{16l^2}\left[\pi x - l\sin\left(\frac{\pi x}{l}\right)\right] \qquad \text{(all displacements are negative!)}$$

Points on the beam axis will move to the left due to buckling. The ten concentrated loads that act in the negative *x*-direction will thus perform positive work:

$$A_{uitw} = \sum_{i=1}^{10}(-F)\times u\left(\tfrac{i}{10}l\right) \tag{1}$$

This work must be equal to the strain energy stored due to bending:

$$\Delta E_v = \int_0^l \tfrac{1}{2}EI\kappa^2 \mathrm{d}x = \int_0^l \tfrac{1}{2}EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = \frac{C^2\pi^4 EI}{64l^3} \tag{2}$$

Solving this using MAPLE by setting (1) and (2) equal to each other gives us the following:

$$F_k = \frac{0,0716\pi^2 EI}{l^2} \qquad \text{(per floor !)}$$

When applying a lot of concentrated loads this can be schematised as a continuous distributed load. The total load Q is then equal to nF where n is the number of floors. For this problem using $n = 10$ we then find:

$$Q_k = q_k \cdot l = n \cdot F_k = \frac{10\cdot 0,0716\cdot\pi^2 EI}{l^2} = \frac{7,07EI}{l_k^{\,2}} = \frac{\pi^2 EI}{(1,18l)^2}$$

For the total buckling load Q_k on the elevator shaft we can find a buckling length of 1,18 *l*. For a distributed load a solution is known from literature:

$$Q_k = q_k \cdot l = \frac{7,94EI}{l^2} = \frac{\pi^2 EI}{l_k^2} = \frac{\pi^2 EI}{(1,12l)^2}$$

The discrete load with 10 concentrated loads and a buckling length of 1,18 *l* gives a slightly lower total buckling load than the one with a completely distributed load.