# Strain energy for linear elastic bending in arbitrary cross sections

The strain energy per unit length of beam can be obtained from:

$$E_v^* = E_c^* = \tfrac{1}{2} M \cdot \kappa$$

$$\text{with}: M = \begin{bmatrix} M_y \\ M_z \end{bmatrix} \text{ and } \kappa = \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}$$

The strain energy can be described in either terms of strains or in terms of sectional forces (moments).
In case of a linear elasticity :

$$E_v^* = E_c^* = \tfrac{1}{2} M_y \kappa_y + \tfrac{1}{2} M_z \kappa_z$$

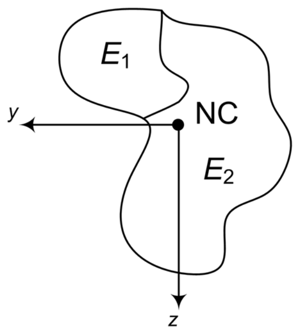


Fig 1 : *Inhomogeneous and unsymmetrical cross section with coordinate system through normal force centre NC.*

Moment and curvature are related to each other through the constitutive matrix of the cross section, definitions used for positive moments and curvatures are presented in figure 2.

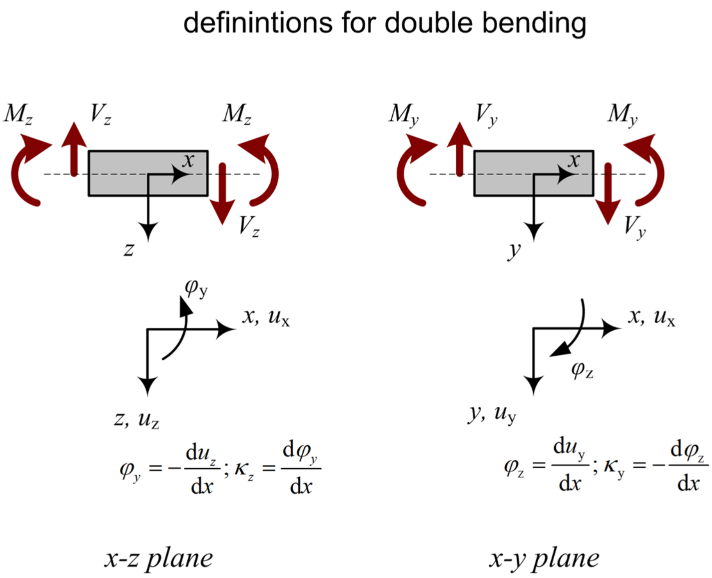


Fig 2 : *Definitions for double bending*

The constitutive matrix is presented in terms of the so-called "*double letter symbols*" with respect to a coordinate system with its origin taken at the normal force centre NC of the cross section as shown in figure 1. In this figure a cross section is shown which consists of two materials with Young's modulus $E_1$ and $E_2$. For such a typical cross section the constitutive matrix of the cross section can be presented as:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \quad \text{and:} \quad \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{EI_{yy} EI_{zz} - EI_{yz}^2} \begin{bmatrix} EI_{zz} & -EI_{yz} \\ -EI_{yz} & EI_{yy} \end{bmatrix} \begin{bmatrix} M_y \\ M_z \end{bmatrix}$$

with:

$$EI_{yy} = \int_A E(y,z) y^2 \mathrm{d}A \qquad EI_{yz} = EI_{zy} = \int_A E(y,z) yz \mathrm{d}A \qquad EI_{zz} = \int_A E(y,z) z^2 \mathrm{d}A$$

The energy can be expressed in terms of the deformation (curvature) (1) or in terms of the generalised stresses (bending moment) (2). Exploring (1) yields to:

$$E_v^* = \tfrac{1}{2} EI_{yy} \kappa_y \kappa_y + \tfrac{1}{2} EI_{yz} \kappa_z \kappa_y + \tfrac{1}{2} EI_{yz} \kappa_y \kappa_z + \tfrac{1}{2} EI_{zz} \kappa_z \kappa_z$$

$$E_v^* = \tfrac{1}{2} EI_{yy} \kappa_y^2 + EI_{yz} \kappa_y \kappa_z + \tfrac{1}{2} EI_{zz} \kappa_z^2$$

Exploring (2) yields to:

$$E_c^* = \frac{1}{EI_{yy} EI_{zz} - EI_{yz}^2} \left[ \tfrac{1}{2} EI_{zz} M_y M_y - \tfrac{1}{2} EI_{yz} M_z M_y - \tfrac{1}{2} EI_{yz} M_y M_z + \tfrac{1}{2} EI_{yy} M_z M_z \right]$$

$$E_c^* = \frac{1}{EI_{yy} EI_{zz} - EI_{yz}^2} \left[ \tfrac{1}{2} EI_{zz} M_y^2 - EI_{yz} M_y M_z + \tfrac{1}{2} EI_{yy} M_z^2 \right]$$

In case the coordinate system coincides with *the principal axes* we obtain well known expressions:

$$E_v^* = \tfrac{1}{2} EI_{yy} \kappa_y^2 + \tfrac{1}{2} EI_{zz} \kappa_z^2 \quad \text{and} \quad E_c^* = \frac{M_y^2}{2EI_{yy}} + \frac{M_z^2}{2EI_{zz}}$$