Lecture 2 : Castigliano's Energy Theorems

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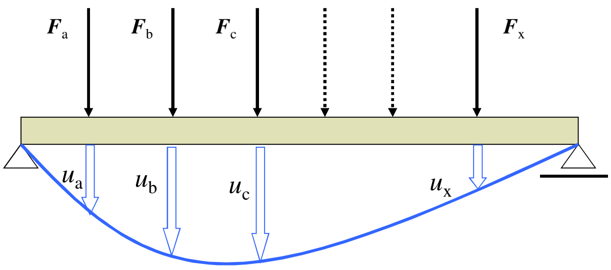
http://icozct.tudelft.nl/TUD_CT/

Castigliano



$$A_{ext} = \tfrac{1}{2} F_a u_a + \tfrac{1}{2} F_b u_b + \tfrac{1}{2} F_c u_c + \ldots + \tfrac{1}{2} F_x u_x$$

Differentiate the Work to a specific force, e.g. F_x:

$$\frac{\partial A_{ext}}{\partial F_x} = \tfrac{1}{2} F_a \frac{\partial u_a}{\partial F_x} + \tfrac{1}{2} F_b \frac{\partial u_b}{\partial F_x} + \tfrac{1}{2} F_c \frac{\partial u_c}{\partial F_x} + \dots$$

$$+ \tfrac{1}{2} u_x + \tfrac{1}{2} F_x \frac{\partial u_x}{\partial F_x}$$

Use Maxwell's notation :

$$u_a = c_{aa} F_a + c_{ab} F_b + c_{ac} F_c + \dots + c_{ax} F_x$$
$$u_b = c_{ba} F_a + c_{bb} F_b + c_{bc} F_c + \dots + c_{bx} F_x$$
$$u_c = c_{ca} F_a + c_{cb} F_b + c_{cc} F_c + \dots + c_{cx} F_x$$
$$u_x = c_{xa} F_a + c_{xb} F_b + c_{xc} F_c + \dots + c_{xx} F_x$$

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Elaborate for all …

$$\frac{\partial u_a}{\partial F_x} = \frac{\partial (c_{aa} F_a + c_{ab} F_b + c_{ac} F_c + \dots + c_{ax} F_x)}{\partial F_x} = c_{ax}$$

$$\frac{\partial u_b}{\partial F_x} = \frac{\partial (c_{ba} F_a + c_{bb} F_b + c_{bc} F_c + \dots + c_{bx} F_x)}{\partial F_x} = c_{bx}$$

$$\frac{\partial u_c}{\partial F_x} = \frac{\partial (c_{ca} F_a + c_{cb} F_b + c_{cc} F_c + \dots + c_{cx} F_x)}{\partial F_x} = c_{cx}$$

$$\frac{\partial u_x}{\partial F_x} = \frac{\partial (c_{xa} F_a + c_{xb} F_b + c_{xc} F_c + \dots + c_{xx} F_x)}{\partial F_x} = c_{xx}$$

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Result:

$$\frac{\partial A_{ext}}{\partial F_x} = \tfrac{1}{2}F_a c_{ax} + \tfrac{1}{2}F_b c_{bx} + \tfrac{1}{2}F_c c_{cx} + ... + \tfrac{1}{2}F_x c_{xx} + \tfrac{1}{2}u_x$$

Maxwell :

$$c_{ax} = c_{xa}$$
$$c_{bx} = c_{xb}$$
$$c_{cx} = c_{xc}$$

$$\frac{\partial A_{ext}}{\partial F_x} = \underbrace{\tfrac{1}{2}c_{xa}F_a + \tfrac{1}{2}c_{xb}F_b + \tfrac{1}{2}c_{xc}F_c + ... + \tfrac{1}{2}c_{xx}F_x}_{\tfrac{1}{2}u_x} + \tfrac{1}{2}u_x$$

Castigliano's 2nd theorem

$$\frac{\partial A_{ext}}{\partial F_x} = u_x$$

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Castigliano

- Work done by external loads A_{ext} is stored in the strain energy E_c (complementary)
- Differentiate the strain energy to a force at location *x* to find the displacement *u* at *x*

$$\frac{\partial E_c}{\partial F_x} = u_x$$

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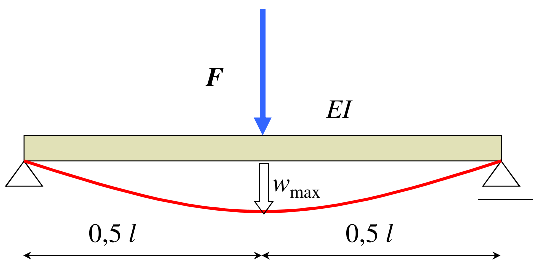
Example



Strain Energy in terms of the load

$$M(x) = \tfrac{1}{2} Fx \quad 0 \leq x \leq \tfrac{1}{2} l$$

$$E_c = \int_o^l \frac{M(x)^2}{2EI} \mathrm{d}x = 2\int_0^{\frac{1}{2}l} \frac{\frac{1}{4} F^2 x^2}{2EI} \mathrm{d}x = \frac{F^2 l^3}{96EI}$$

$$w = \frac{\mathrm{d}E_c}{\mathrm{d}F} = \frac{2Fl^3}{96EI} = \frac{Fl^3}{48EI}$$

Steps of the solution strategy

- Castigliano : differentiate E_C to a force.
- E_C is integral of the square of the moment distribution
- so

…. differentiate after integration …

IS THIS SMART TO DO ??

Smart help

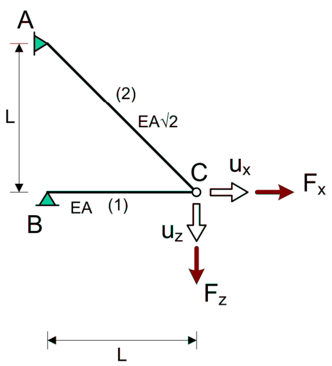
$$w = \int_0^l \frac{m(x) \times M(x)}{EI} \mathrm{d}x$$

$$w = \int_0^l \frac{2M(x)}{2EI} \times \frac{\mathrm{d}M(x)}{\mathrm{d}F} \mathrm{d}x = \int_0^l \frac{M(x)}{EI} \times \boxed{\frac{\mathrm{d}M(x)}{\mathrm{d}F}} \mathrm{d}x$$

Castigliano

- Second theorem $$\frac{\partial E_c}{\partial F_x} = u_x$$
- First theorem $$\frac{\partial E_v}{\partial u_x} = F_x$$

Application



$$\frac{\partial E_v}{\partial u_x} = F_x$$

(energy expressed in u_x and u_z)

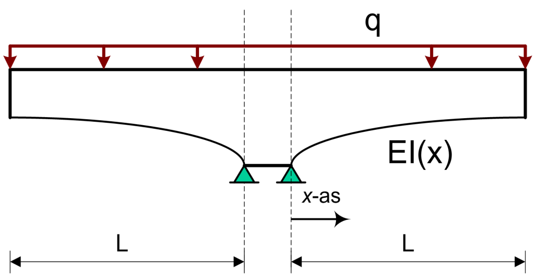
$$\frac{\partial E_c}{\partial F_x} = u_x$$

(energy expressed in F_x and F_z)

Example

deformation with Castigliano:

- Find the displacement at the free end
- Find the rotation at the free end



Given :

$$EI(x) = EI_o\left(1-\frac{x}{l}\right); \quad EI_o = 150\times10^6 \text{ kNm}^2; \quad l = 40 \text{ m}; \quad q = 35 \text{ kN/m}$$

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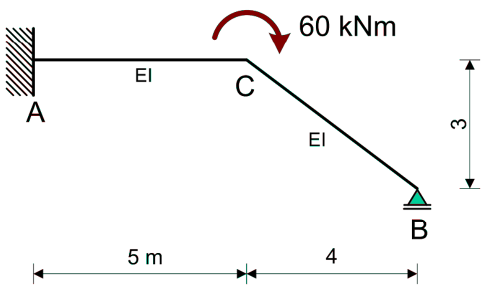
Example

Force distribution with Castigliano:

- Find the support reaction at B using Castigliano
- Draw the moment distribution



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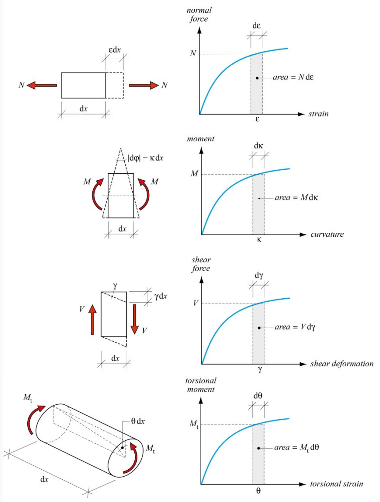
Application : shear stiffness



Beam:

- Axial loading
- Bending
- Shear
- Torsion

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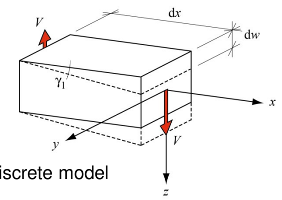
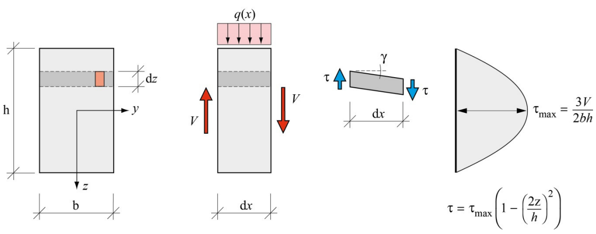
Asumme LE, find *k*



discrete model

$$E_v = E_c = \tfrac{1}{2} V \times \mathrm{d}w = \tfrac{1}{2} V \times \gamma_1 \times \mathrm{d}x$$

$$\text{with}: \; V = k \times \gamma_1$$



continuous model

- Shear stress distribution not constant over depth
- If LE then shear strain γ not constant!

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Equal amount of stored energy

discrete :

$$E_c = \tfrac{1}{2} V \times \mathrm{d}w = \tfrac{1}{2} V \times \gamma_1 \times \mathrm{d}x$$

continuous :

$$E_c = \int_{z=-h/2}^{z=h/2} \frac{\tau(z)^2}{2G} b\mathrm{d}x\mathrm{d}z$$

$$\text{with} : \tau(z) = \frac{V \times S_z^{(a)}}{b \times I_{zz}} = \frac{6V\left(\frac{h^2}{4} - z^2\right)}{bh^3}$$

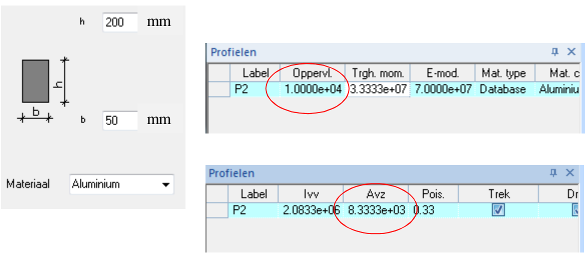
$$\int_{z=-h/2}^{z=h/2} \frac{\tau(z)^2}{2G} b\mathrm{d}x\mathrm{d}z = \tfrac{1}{2} V \gamma_1 \mathrm{d}x$$

$$\frac{b}{G} \int_{z=-h/2}^{z=h/2} \tau(z)^2 \mathrm{d}z = V \gamma_1$$

$$\frac{6V^2}{5Gbh} = V\gamma_1 \Rightarrow \quad \frac{V}{\gamma_1} = \tfrac{5}{6} Gbh$$

$$k = \tfrac{5}{6} GA \text{ with: } G = \frac{E}{2(1+\nu)}$$

Analysis Software



$$A = b \times h = 200 \times 50 = 10000 \text{ mm}^2$$

$$A_{shear} = \tfrac{5}{6} \times 10000 = 8333,33 \text{ mm}^2$$

$$G = \frac{E}{2(1+\nu)} \quad \text{so:} \quad GA_{shear} = \frac{GA}{\eta} \quad \text{with } \eta = \tfrac{6}{5}$$

Beam model with shear deformation

- Euler-Bernoulli – no shear deformation (one ODE)

$$EI\frac{d^4w}{dx^4} = q(x)$$

- Timoshenko beam theory – bending and shear (two coupled ODE's)

$$EI\frac{d^2\varphi}{dx^2} - GA_{shear}\left(\frac{dw}{dx}+\varphi\right) = 0 \quad (1)$$

$$GA_{shear}\left(\frac{d^2w}{dx^2}+\frac{d\varphi}{dx}\right) = -q \quad (2)$$

CIE4190 Analysis of Slender Structures

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