Lecture 3 :

Potential Energy

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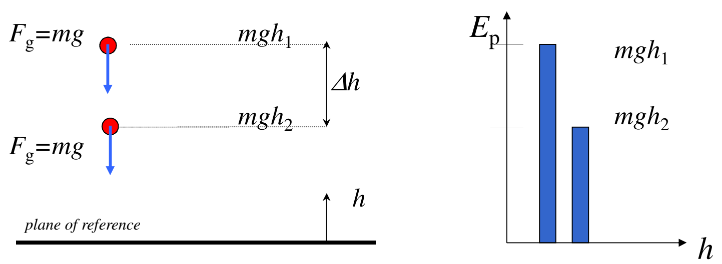
Potential Energy

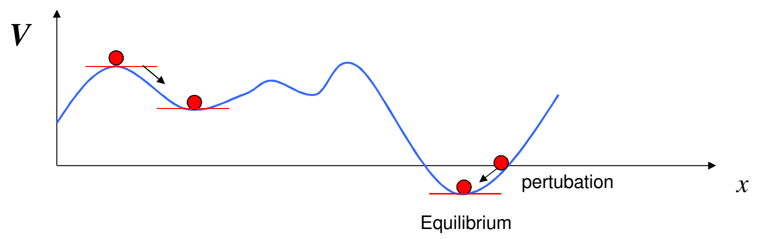


no kinetic energy (statics)

Total Energy

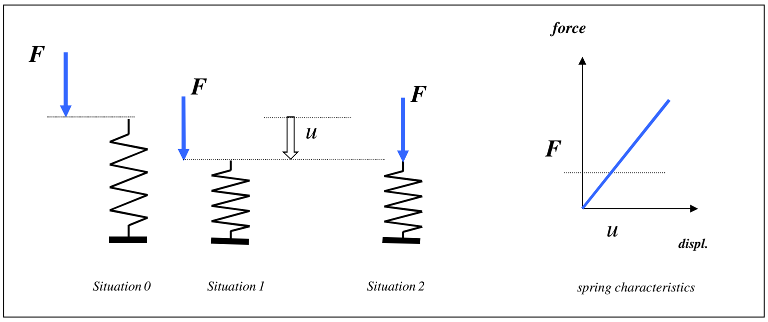
- Sum of all energy in a system is constant
- Sum of all Potential energy = C
- Potential energy:
  - Loads (energy with respect to reference, reduces)
  - Strain energie (E_v, increases)

Stable Equilibrium



Stationary Energy Function
(hor. tangent)

Potential Energy due to Loads
Load Potential



$$V = E_{\mathrm{v}} - \Delta E_{\mathrm{p}} = \tfrac{1}{2} \times k \times u^2 - F \times u$$

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Stationary Energy Function at a Minimum Energy level

- Extreme = derivative with respect to a state variable (***u***) must be zero
- Extreme is a minimum = 2nd deriv > 0

$$\frac{\mathrm{d}V}{\mathrm{d}u} = k \times u - F = 0 \quad \textit{and} \quad \frac{\mathrm{d}^2V}{\mathrm{d}u^2} = k \quad (\; > 0 \text{ is minimum})$$

principle of minimum potential energy

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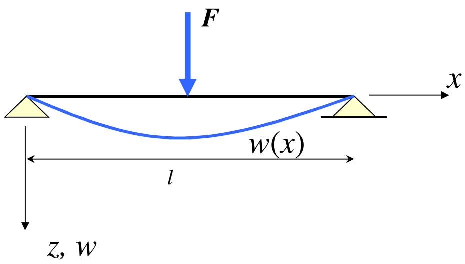
Application

- Approximate displacement field
- Demand Stationary Potential Energy:

  derivative(s) of ***V*** with respect to all state variables a_i must be zero.

$$\delta V = \frac{\partial V}{\partial a_1}\delta a_1 + \frac{\partial V}{\partial a_2}\delta a_2 + \ldots + \frac{\partial V}{\partial a_i}\delta a_i = 0$$

Example : Beam



$$w = a \sin\left(\frac{\pi x}{l}\right)$$

Solution

$$\sin^2 a = \tfrac{1}{2} - \tfrac{1}{2}\cos 2a$$

$$V = \tfrac{1}{2} EI \int_0^l w''^2 \mathrm{d}x - Fa = \tfrac{1}{2}\frac{\pi^4 EIa^2}{l^4} \int_0^l \sin^2\left(\frac{\pi x}{l}\right) \mathrm{d}x - Fa$$

$$V = \tfrac{1}{2}\frac{\pi^4 EIa^2}{l^4} \int_0^l \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{l}\right) \mathrm{d}x - Fa = \tfrac{1}{2}\frac{\pi^4 EIa^2}{l^4} \times \tfrac{1}{2} l - Fa$$

$$V = \frac{\pi^4 EIa^2}{4l^3} - Fa$$

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Minimalise

$$\frac{\mathrm{d}V}{\mathrm{d}a} = \frac{\pi^4 EIa}{2l^3} - F = 0 \quad \Rightarrow$$

$$w_{mid-span} = a = \frac{Fl^3}{\left(\pi^4/2\right)EI} = \frac{Fl^3}{48.705EI}$$

approximation

$$V = -\frac{F^2 l^3}{97.409EI}$$

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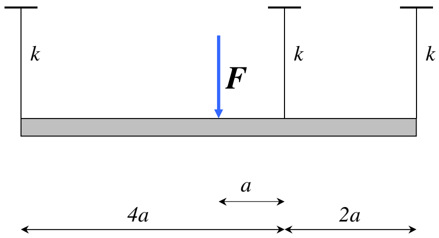
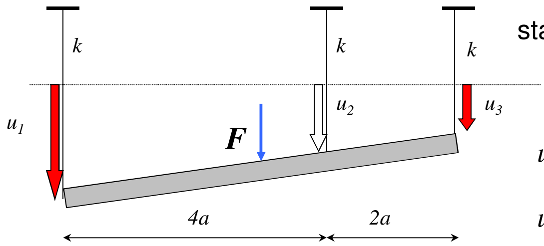
Example : Rigid Block



Displacement field u (assumption)



state variables u_1 and u_3

$$u_2 = u_1 - \left(\frac{u_1 - u_3}{6a} \right) \times 4a$$

$$u_2 = \tfrac{1}{3}(u_1 + 2u_3)$$

$$V = \tfrac{1}{2}ku_1^2 + \tfrac{1}{2}k\left(\tfrac{1}{3}(u_1 + 2u_3)\right)^2 + \tfrac{1}{2}ku_3^2 - F \times \left(\tfrac{1}{2}(u_1 + u_3)\right) \Leftrightarrow$$

$$V = \tfrac{10}{18}ku_1^2 + \tfrac{4}{18}ku_1u_3 + \tfrac{13}{18}ku_3^2 - \tfrac{1}{2}F\left(u_1 + u_3\right)$$

$$\delta V = \frac{\partial V}{\partial u_1}\delta u_1 + \frac{\partial V}{\partial u_3}\delta u_3 = 0 \longrightarrow \frac{\partial V}{\partial u_1} = 0; \quad \frac{\partial V}{\partial u_3} = 0$$

Result

$$\tfrac{20}{18}ku_1 + \tfrac{4}{18}ku_3 - \tfrac{1}{2}F = 0$$
$$\tfrac{4}{18}ku_1 + \tfrac{26}{18}ku_3 - \tfrac{1}{2}F = 0 \qquad \Rightarrow u_3 = \tfrac{8}{11}u_1;$$

$$u_1 = \tfrac{11}{28}\frac{F}{k} \qquad u_2 = \tfrac{9}{28}\frac{F}{k} \qquad u_3 = \tfrac{8}{28}\frac{F}{k}$$

exact solution

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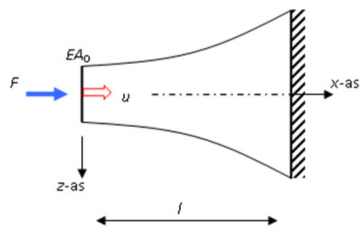
Assignment



$$EA(x) = \frac{2EA_o}{2 - \dfrac{x}{l}}$$

Assume a displacement field:

$$u(x) = a\left(1 - \frac{x}{l}\right)$$

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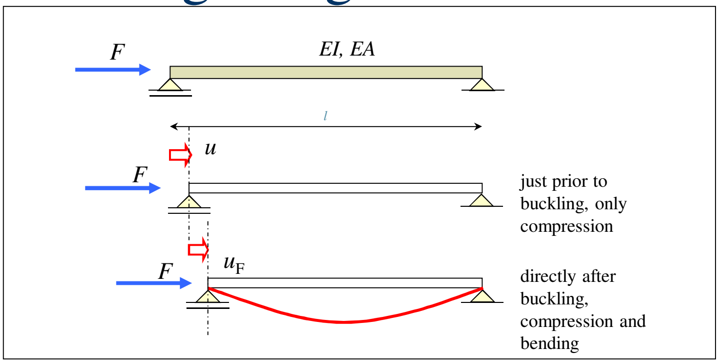
Application Potential Energy Buckling … again



assume: $w = a \sin\left(\dfrac{\pi x}{l}\right)$

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Stationary potential energy:

$$V = \tfrac{1}{2} EI \int_0^l w''^2 \mathrm{d}x - F u_F = \tfrac{1}{2} EI \int_0^l w''^2 \mathrm{d}x - F \, \tfrac{1}{2} \int_0^l (w')^2 \mathrm{d}x$$

$$V = \frac{\pi^4 EIa^2}{4l^3} - \frac{\pi^2 Fa^2}{4l}$$

$$\frac{\mathrm{d}V}{\mathrm{d}a} = \frac{\pi^4 EIa}{2l^3} - \frac{\pi^2 Fa}{2l} = 0 \quad \Rightarrow \quad \boxed{F_k = \frac{\pi^2 EI}{l^2}}$$

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> restart;
> w:=a*sin(Pi*x/L);
> V:=EI*int((1/2)*diff(w,x$2)^2,x=0..L)-F*int((1/2)*diff(w,x)^2,x=0..L);
> eq:=diff(V,a)=0;
> solve(eq,F);
```

What about a?

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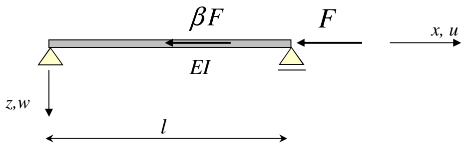
Assigment, F_{k} with Pot. Energy?



Compressive element with
additonal loading at mid span

<u>Approach Potential Energy: (versus Rayleigh method)</u>

- Assume a kinematically admissible displacement field
- Potential energy due to deformation
- Potential energy due to loads
- Apply min Pot. Energy principle

answer :

$$F_k = \frac{2\pi^2 EI}{(\beta+2)l^2}$$

$$F_k = \alpha\frac{\pi^2 EI}{l^2} \quad \text{with:} \quad \alpha = \frac{2}{\beta+2}$$

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