Write on all your pages:
STUDENTNUMBER :
NAME :

Exam

CTB3330/CIE3109-09

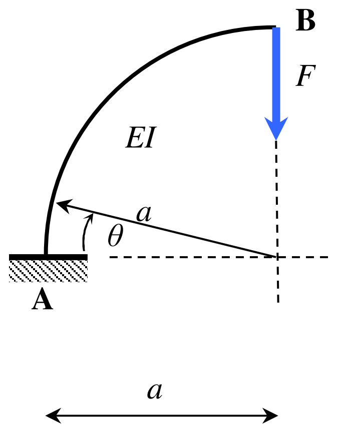
STRUCTURAL MECHANICS 4

1 April 2014,

- Example question on work & Energy.
- Answer every problem on a **separate sheet**.
- Write your name and studentnumber on every sheet
- In the assessment of your work the quality of presentation is also considered
- The use of mobile phones is not allowed, so turn them off and put them away!
- Use the sheet with formulas at the end of this exam.
- Pay attention to the time given for every problem.

PROBLEM 2 : Work and Energy (ca 40 min)

A prismatic curved beam with radius a is loaded at B by a vertical concentrated load F. The bending stiffness is denoted with EI and the axial deformation is neglected.



Remark : Make use of the given formula sheet if necessary!!!

Questions:

a) Sketch the moment distribution (*M-line*) including the deformation sign and the extreme value of the bending moment expressed in F and a.

b) Find the strain energy stored in this structure by bending.
HINT: Use polar coordinates

c) Find the vertical displacement at B using Castigliano's theorem expressed in F, a and EI.

FORMULA SHEET

Inhomogeneous and/or non-symmetrical cross sections:

$$\varepsilon(y,z)=\varepsilon+\kappa_y y+\kappa_z z \qquad \text{en:} \quad \sigma(y,z)=E(y,z)\times\varepsilon(y,z)$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix}=\begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix}\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \quad \text{en} \quad \begin{bmatrix} e_y \\ e_z \end{bmatrix}=\frac{1}{EA}\begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix}\begin{bmatrix} -1/y_1 \\ -1/z_1 \end{bmatrix}$$

$$s_x^{(a)}=-\frac{V_y ES_y^{(a)}}{EI_{yy}}-\frac{V_z ES_z^{(a)}}{EI_{zz}}; \quad \text{of:} \quad s_x^{(a)}=-\frac{R_M^{(a)}}{M}V; \quad \sigma_{xt}=\frac{s_x^{(a)}}{b^{(a)}}$$

$$\tan(2\alpha)=\frac{2EI_{yz}}{\left(EI_{yy}-EI_{zz}\right)}; \quad EI_{1,2}=\tfrac{1}{2}\left(EI_{yy}+EI_{zz}\right)\pm\sqrt{\left(\tfrac{1}{2}\left(EI_{yy}-EI_{zz}\right)\right)^2+EI_{yz}^2}$$

Deformation energy:

$$E_v=\int \tfrac{1}{2}EA\varepsilon^2 \mathrm{d}x \quad \text{(extension)}$$

$$E_v=\int \tfrac{1}{2}EI\kappa^2 \mathrm{d}x \quad \text{(bending)}$$

Kinematic relations:

$$\varepsilon=\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\kappa=-\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

Complementary energy:

$$E_c=\int \frac{N^2}{2EA}\mathrm{d}x \quad \text{(extension)}$$

$$E_c=\int \frac{M^2}{2EI}\mathrm{d}x \quad \text{(bending)}$$

Constitutive relations:

$$N=EA.\varepsilon$$

$$M=EI.\kappa$$

Castigliano's theoreme:

$$F_i=\frac{\partial E_v}{\partial u_i} \qquad u_i=\frac{\partial E_c}{\partial F_i}$$

Work method with unit load:

$$u=\int \frac{M(x)m(x)dx}{EI}$$

Rayleigh:

$$F_k=\frac{E_v}{\int \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x}$$

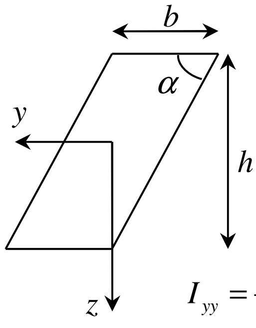
Mathematical tricks:

$$\int \cos^2 x\,\mathrm{d}x=\tfrac{1}{2}x+\tfrac{1}{2}\cos x\sin x$$

$$\int \sin^2 x\,\mathrm{d}x=\tfrac{1}{2}x-\tfrac{1}{2}\cos x\sin x$$

$$\int \cos x\sin x\,\mathrm{d}x=-\tfrac{1}{2}\cos^2 x$$

$$\cot x=\frac{1}{\tan x}$$

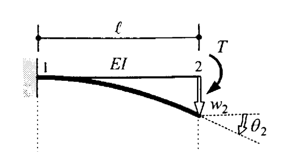
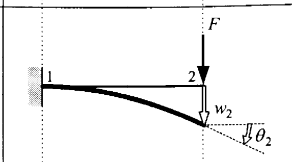
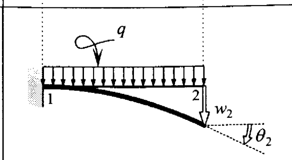


$$I_{yy}=\tfrac{1}{12}b^3h+\tfrac{1}{12}bh^3\cot^2\alpha$$

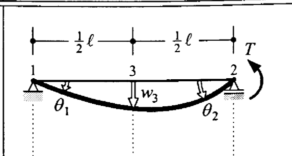
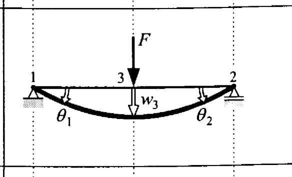
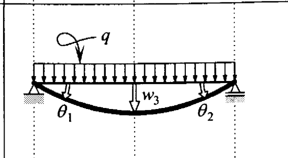
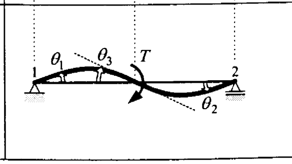
$$I_{yz}=\tfrac{1}{12}bh^3\cot\alpha$$

$$I_{zz}=\tfrac{1}{12}bh^3$$

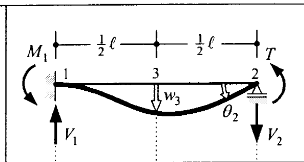
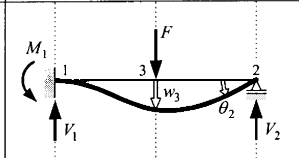
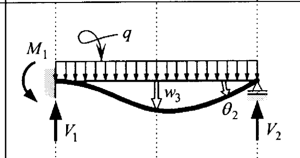
vergeet-mij-nietjes

(1)		$\theta_2 = \frac{T\ell}{EI}$; $w_2 = \frac{T\ell^2}{2EI}$
(2)		$\theta_2 = \frac{F\ell^2}{2EI}$; $w_2 = \frac{F\ell^3}{3EI}$
(3)		$\theta_2 = \frac{q\ell^3}{6EI}$; $w_2 = \frac{q\ell^4}{8EI}$

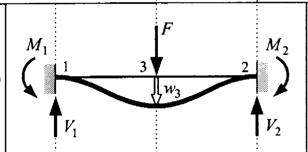
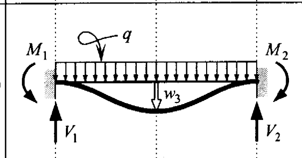
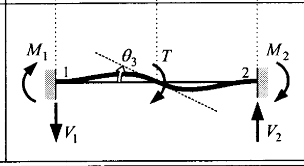
vrij opgelegde ligger (statisch bepaald)

(4)		$\theta_1 = \frac{1}{6}\frac{T\ell}{EI}$; $\theta_2 = \frac{1}{3}\frac{T\ell}{EI}$; $w_3 = \frac{1}{16}\frac{T\ell^2}{EI}$
(5)		$\theta_1 = \theta_2 = \frac{1}{16}\frac{F\ell^2}{EI}$; $w_3 = \frac{1}{48}\frac{F\ell^3}{EI}$
(6)		$\theta_1 = \theta_2 = \frac{1}{24}\frac{q\ell^3}{EI}$; $w_3 = \frac{5}{384}\frac{q\ell^4}{EI}$
(a)		$\theta_1 = \theta_2 = \frac{1}{24}\frac{T\ell}{EI}$; $\theta_3 = \frac{1}{12}\frac{T\ell}{EI}$; $w_3 = 0$

statisch onbepaalde ligger (enkelzijdig ingeklemd)

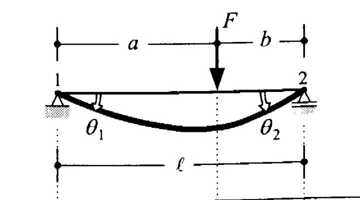
(7)		$\theta_2 = \frac{1}{4}\frac{T\ell}{EI}$; $w_3 = \frac{1}{32}\frac{T\ell^2}{EI}$ $M_1 = \frac{1}{2}T$; $V_1 = V_2 = \frac{3}{2}\frac{T}{\ell}$
(8)		$\theta_2 = \frac{1}{32}\frac{F\ell^2}{EI}$; $w_3 = \frac{7}{768}\frac{F\ell^3}{EI}$ $M_1 = \frac{3}{16}F\ell$; $V_1 = \frac{11}{16}F$; $V_2 = \frac{5}{16}F$
(9)		$\theta_2 = \frac{1}{48}\frac{q\ell^3}{EI}$; $w_3 = \frac{1}{192}\frac{q\ell^4}{EI}$ $M_1 = \frac{1}{8}q\ell^2$; $V_1 = \frac{5}{8}q\ell$; $V_2 = \frac{3}{8}q\ell$

statisch onbepaalde ligger (tweezijdig ingeklemd)

(10)		$w_3 = \frac{1}{192}\frac{F\ell^3}{EI}$ $M_1 = M_2 = \frac{1}{8}F\ell$; $V_1 = V_2 = \frac{1}{2}F$
(11)		$w_3 = \frac{1}{384}\frac{q\ell^4}{EI}$ $M_1 = M_2 = \frac{1}{12}q\ell^2$; $V_1 = V_2 = \frac{1}{2}q\ell$
(b)		$\theta_3 = \frac{1}{16}\frac{T\ell}{EI}$; $w_3 = 0$ $M_1 = M_2 = \frac{1}{4}T$; $V_1 = V_2 = \frac{3}{2}\frac{T}{\ell}$

Enkele formules voor prismatische liggers met buigstijfheid EI.
T, F en q zijn belastingen door resp. een koppel, kracht en gelijkmatig verdeelde belasting.
M_i en V_i zijn het buigend moment en de dwarskracht op einddoorsnede i van de ligger ten gevolge van de oplegreacties.

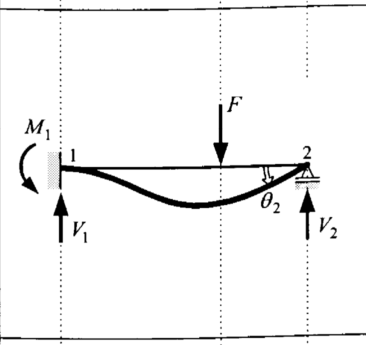
(c)



$$\theta_1 = \frac{Fab(\ell+b)}{6EI\ell} = \frac{F\ell^2}{6EI}\left(2\frac{a}{\ell} - 3\frac{a^2}{\ell^2} + \frac{a^3}{\ell^3}\right)$$

$$\theta_2 = \frac{Fab(\ell+a)}{6EI\ell} = \frac{F\ell^2}{6EI}\left(\frac{a}{\ell} - \frac{a^3}{\ell^3}\right)$$

(d)



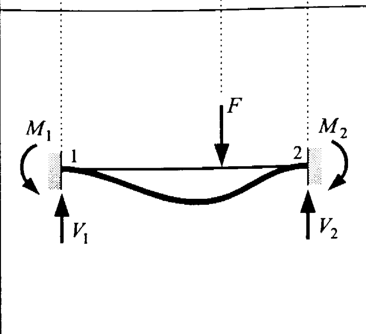
$$M_1 = \frac{Fb(\ell^2 - b^2)}{2\ell^2} = F\ell\left(\frac{a}{\ell} - \frac{3}{2}\frac{a^2}{\ell^2} + \frac{1}{2}\frac{a^3}{\ell^3}\right)$$

$$V_1 = \frac{Fb(3\ell^2 - b^2)}{2\ell^3} = F\left(1 - \frac{3}{2}\frac{a^2}{\ell^2} + \frac{1}{2}\frac{a^3}{\ell^3}\right)$$

$$V_2 = \frac{Fa^2(3\ell - a)}{2\ell^3} = F\left(\frac{3}{2}\frac{a^2}{\ell^2} - \frac{1}{2}\frac{a^3}{\ell^3}\right)$$

$$\theta_2 = \frac{Fa^2b}{4EI\ell} = \frac{F\ell^2}{4EI}\left(\frac{a^2}{\ell^2} - \frac{a^3}{\ell^3}\right)$$

(e)



$$M_1 = \frac{Fab^2}{\ell^2} = F\ell\left(\frac{a}{\ell} - 2\frac{a^2}{\ell^2} + \frac{a^3}{\ell^3}\right)$$

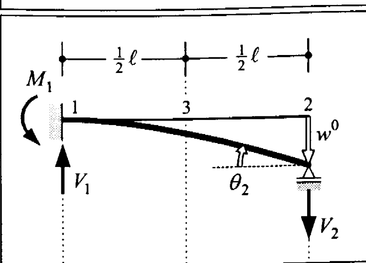
$$V_1 = \frac{Fb^2(\ell + 2a)}{\ell^3} = F\left(1 - 3\frac{a^2}{\ell^2} + 2\frac{a^3}{\ell^3}\right)$$

$$M_2 = \frac{Fa^2b}{\ell^2} = F\ell\left(\frac{a^2}{\ell^2} - \frac{a^3}{\ell^3}\right)$$

$$V_2 = \frac{Fa^2(\ell + 2b)}{\ell^3} = F\ell\left(3\frac{a^2}{\ell^2} - 2\frac{a^3}{\ell^3}\right)$$

drie bij-de handjes

(f)

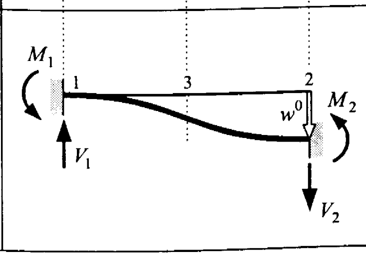


$$M_1 = \frac{3EI}{\ell^2}w^0; \quad V_1 = V_2 = \frac{3EI}{\ell^3}w^0$$

$$\theta_2 = \frac{3}{2}\frac{w^0}{\ell}$$

$$\theta_3 = \frac{9}{8}\frac{w^0}{\ell}; \quad w_3 = \frac{5}{16}w^0$$

(g)



$$M_1 = M_2 = \frac{6EI}{\ell^2}w^0; \quad V_1 = V_2 = \frac{12EI}{\ell^3}w^0$$

$$\theta_3 = \frac{3}{2}\frac{w^0}{\ell}; \quad w_3 = \frac{1}{2}w^0$$

zettingen

ANSWER MODEL

Problem 2

a) Set up the moment distribution of a straight cantilever beam and project the moments to the position on the curved beam and put the value perpendicular to the beam axis.

b) Use the standard expression for the energy stored due to bending. The unit per length of element length is in this case along the curved beam axis:

$$E_v = \int_s \frac{M(s)^2}{2EI}\,\mathrm{d}s$$

The bending moment along the beam's *s*-coordinate can be transferred to a coordinate system in polar coordinates. Use for this:

$$\mathrm{d}s = a\,\mathrm{d}\theta$$

This results in a moment which depends on θ and can be expressed as:

$$M(\theta) = -Fa\cos\theta$$

c) By differentiating the deformation energy to the load, the displacement at the point of application of the (concentrated) load can be obtained (Castigliano's 2nd theorema):

$$E_v = \int_s \frac{M(s)^2}{2EI}\,\mathrm{d}s = \int_{\theta=0}^{\theta=\frac{1}{2}\pi} \frac{\left(Fa\cos\theta\right)^2}{2EI}\,a\,\mathrm{d}\theta$$

Elaborating results in the expression for the displacement:

$$w = \frac{\mathrm{d}E_v}{\mathrm{d}F} = \frac{\pi F a^3}{4EI}$$