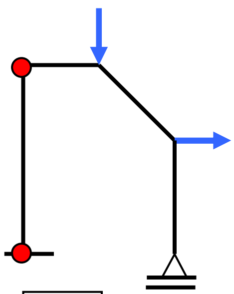
ANSWERS

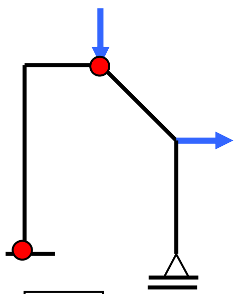
Problem 1 : Plasticity

a) The given structure is statically indeterminate with a degree of one. To obtain a mechanism, two additional hinges are required. Hinges can occur at four positions. Thus, six mechanisms have to be investigated. However, moments can not occur at E due to the loading and support conditions at B thus only three mechanisms are left to be considered. Possible mechanisms are shown below. Each mechanism is shown in detail on the following pages.



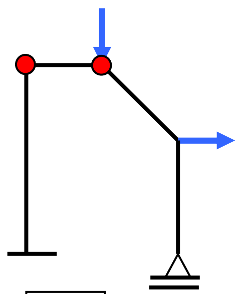
1

$$F_p = \frac{4M_p}{5a}$$



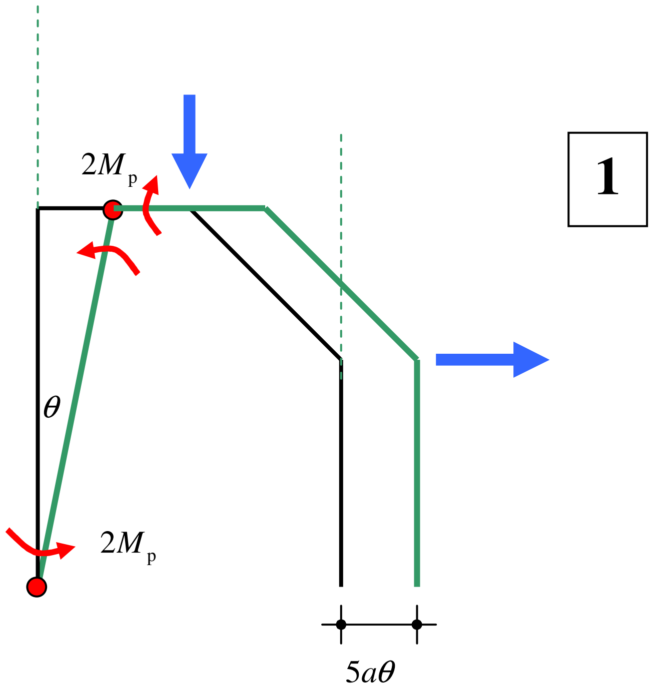
2

$$F_p = \frac{4M_p}{9a}$$



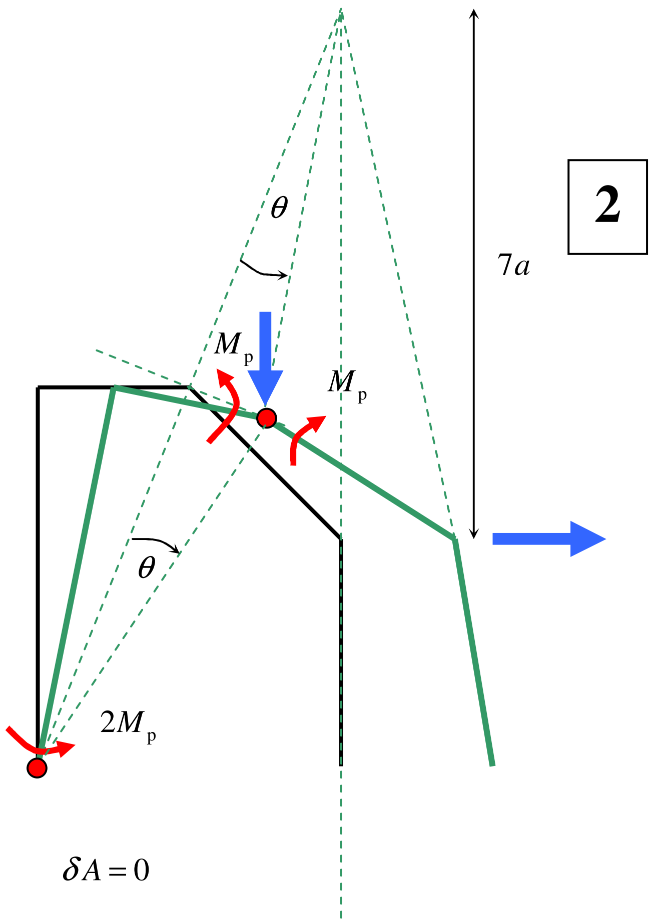
3

$$F_p = \frac{M_p}{a}$$



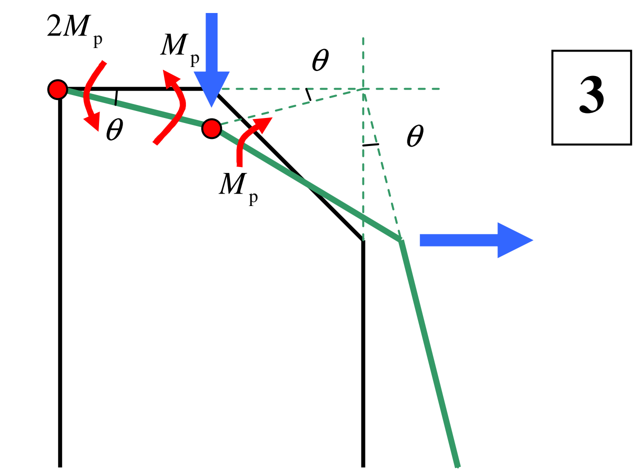
$\delta A = 0$

$$-2M_p \times \delta\theta - 2M_p \times \delta\theta - 2M_p \times 0(!) + F_p \times 5a\delta\theta = 0 \quad \Leftrightarrow \quad F_p = \frac{4M_p}{5a}$$



$\delta A = 0$

$$-2M_p \times \delta\theta - M_p \times \delta\theta - M_p \times \delta\theta + F_p \times 2a\delta\theta + F_p \times 7a\delta\theta = 0 \quad \Leftrightarrow \quad F_p = \frac{4M_p}{9a}$$

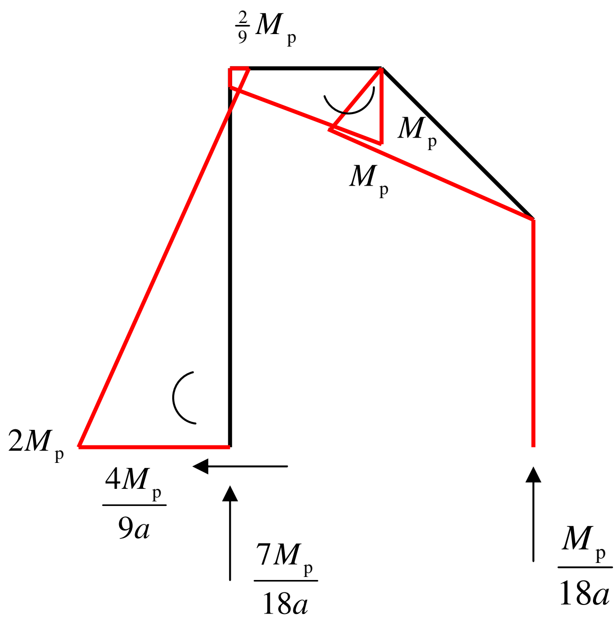


$$\delta A = 0$$

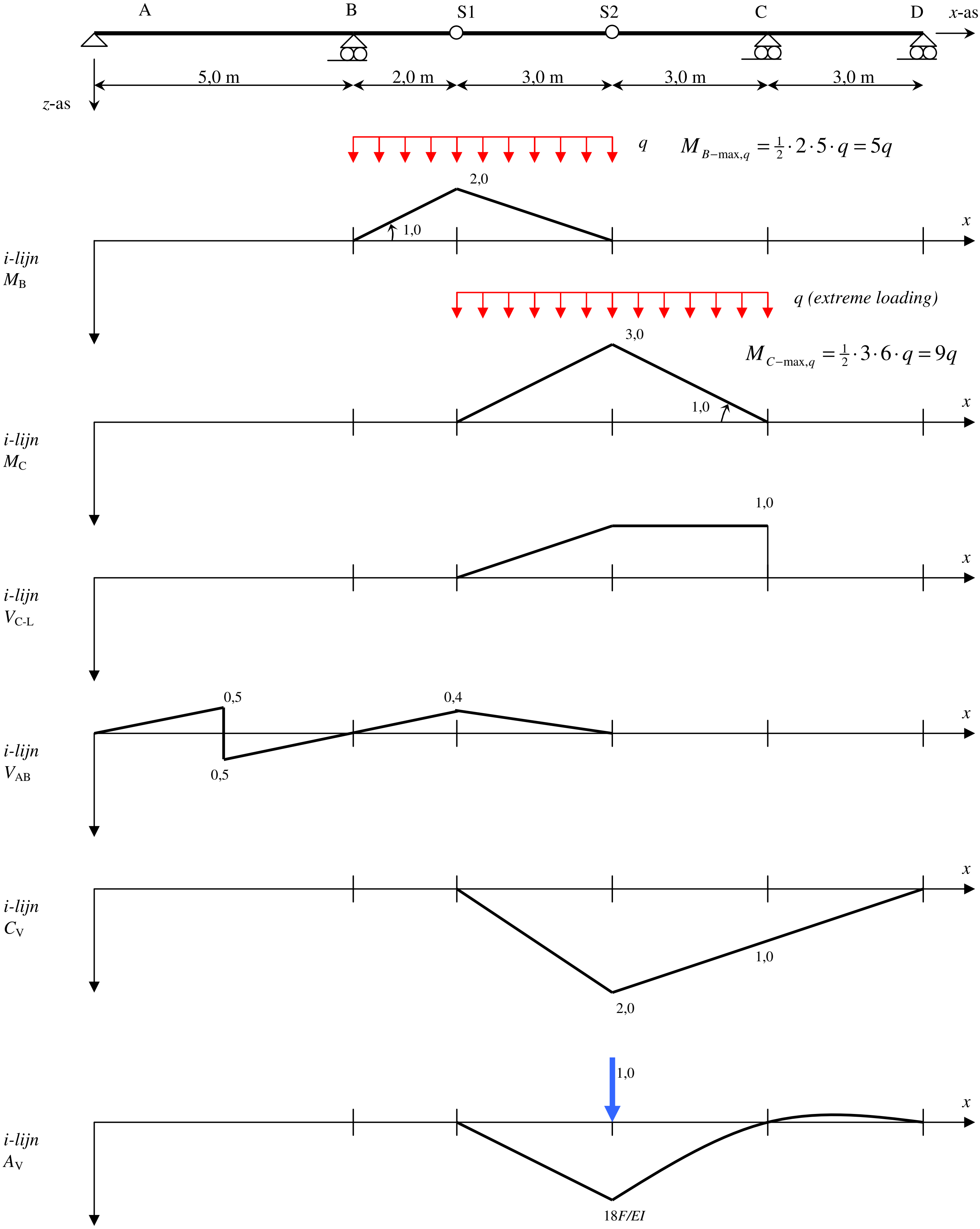
$$-2M_p \times \delta\theta - M_p \times \delta\theta - M_p \times \delta\theta + F_p \times 2a\delta\theta + F_p \times 2a\delta\theta = 0 \quad \Leftrightarrow F_p = \frac{M_p}{a}$$

b) The ultimate load is the lowest load found: $F_p = \dfrac{4M_p}{9a} = 0,444\dfrac{M_p}{a}$

c) To find the correct moment distribution start with the support reactions. The moment distribution shows a statically admissible distribution in which at no cross section the bending moments exceeds the utltimate plastic capacity of the cross section. (**uniqueness theorem**)

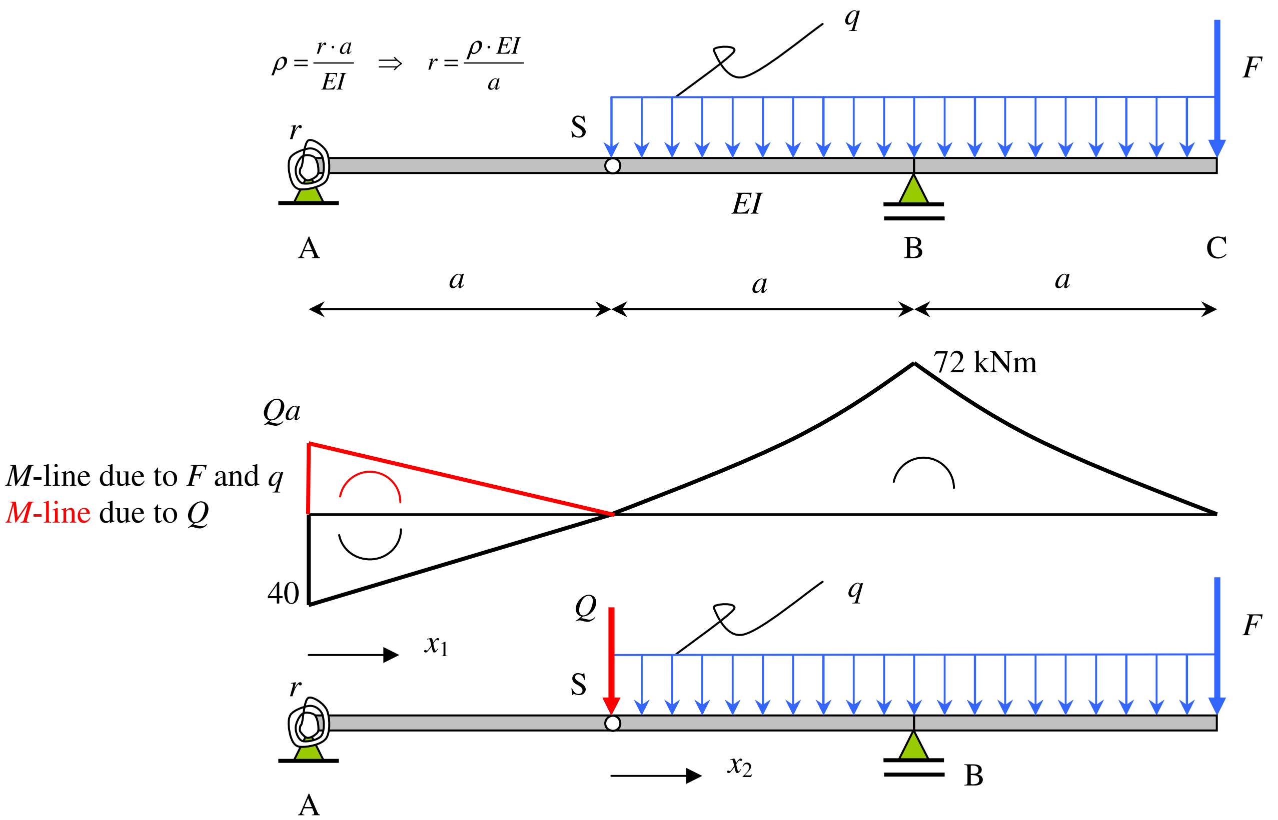


Problem 2 : Influence lines



Problem 3 : Work and Energy Methods

a) The moment distribution can be found base don equilibrium. The distributed load q is in fact balanced thus only the concentrated load is contributing to the moment in the spring and the member AS. (first years knowledge !!)



b) To find the displacement at S a dummy load Q is required. In red the moment distribution due to the dummy load is presented. Since the moment distribution over SB and BC are exhibiting mirror symmetry, only part SB has to be examined. Using a local coordinate for part AS and SB denoted with x_1 and x_2, the deformation energy can easily be expressed in terms of F, q and Q :

$$E_v = \frac{M_v^2}{2r} + \int_{x_1=0}^{x_1=a} \frac{\left((F-Q)(a-x_1)\right)^2}{2EI} dx_1 + 2\int_{x_2=0}^{x_2=a} \frac{\left(Fx_2 + \frac{1}{2}qx_2^2\right)^2}{2EI} dx_2 \qquad \text{met:}$$

$$M_v = (F-Q)\cdot a$$

Using Castigliano's theorema solves the deflection at S: (note: 2nd integral is irrelevant)

$$w_S = \frac{\partial E_v}{\partial Q} = -\frac{(F-Q)a^3}{\rho EI} - \frac{(F-Q)a^3}{3EI}$$

The dummy load is zero which results in: $w_S = -\frac{Fa^3(3+\rho)}{3\rho EI} = -\frac{Fa^3}{\rho EI} - \frac{Fa^3}{3EI}$

Hinge S will move upwards due tot the load F.

c) Using the given values results in a displacement of 0,0533 m upwards. This displacement is independent of the distributed load q!

d) The total deformation energy stored due to the loading is: (note : 2nd integral is essential)

$$E_v = \frac{M_v^2}{2r} + \int_{x_1=0}^{x_1=a} \frac{\left(F\left(a-x_1\right)\right)^2}{2EI} dx_1 + 2\int_{x_2=0}^{x_2=a} \frac{\left(Fx_2 + \frac{1}{2}qx_2^2\right)^2}{2EI} dx_2$$

$$E_v = \frac{a^3\left(10F^2(1+\rho) + qa\rho(5F + qa\rho)\right)}{20\rho EI}$$

e) If the spring stiffness becomes very small, point A turns into a hinge. A mechanism will occur. The load can move infinitely and thus produce infinite work. The bending deformation will be small compared to the deformation energy stored in the spring. So practically all deformation energy has to be taken by the spring(!) which results in hardly any deformation energy (and curvature) in the elements loaded in bending. The bars will remain straight:

$$\underset{\lim \rho \to 0}{E_v} = F \cdot w = \frac{\left(F \cdot a\right)^2}{2r} = \infty$$

Problem 4 : Non-symmetrical cross sections

a) See the lecture notes.
b) The axial stiffness of the cross section can be found with:

$$EA = 2a \times 2a \times E_1 - a \times a \times E_1 + a \times a \times E_2 = 216 \times 10^9$$

The origin of the coordinate system used is located at the NC. The vertical position of the NC with respect to the upper side of the cross section is:

$$\Delta z_{NC} = 400 \text{ mm}$$

The horizontal position with respect to the left side of the cross section is:

$$\Delta y_{NC} = 200 \text{ mm}$$

c) The *cross sectional constitutive relation* relates the sectional forces to the deformations if the cross section. The bending stiffnesses can be found using the strategy outlined in the lecure notes. This example is very basic so only answers are presented here:

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix} \text{ cross sectional constitutive relation [N, mm]}$$

$$K = 10^{12} \begin{bmatrix} 0,216 & 0 & 0 \\ 0 & 4320 & 1080 \\ 0 & 1080 & 4320 \end{bmatrix} \quad \overline{f} = \begin{bmatrix} 0 \\ 0 \\ 1910 \times 10^6 \end{bmatrix}$$

d) Since this structure is loaded in bending only, the strain ε at the NC must be zero. The curvatures can be found with the constitutive relation:

$$\varepsilon = \frac{N}{EA} = 0$$

$$\kappa_y = \frac{1}{EI_{yy} EI_{zz} - EI_{yz}^2} \left(EI_{zz} \times M_y - EI_{yz} \times M_z \right) = -0,1179 \times 10^{-6}$$

$$\kappa_z = \frac{1}{EI_{yy} EI_{zz} - EI_{yz}^2} \left(-EI_{yz} \times M_y + EI_{yy} \times M_z \right) = 0,4716 \times 10^{-6}$$

The direction of the *plane of loading* and the *plane of curvature* can be obtained with:

$$\tan \alpha_m = \frac{M_z}{M_y} \quad \Rightarrow \quad \alpha_{\text{m}} = 90^{\circ}; \qquad \tan \alpha_k = \frac{\kappa_z}{\kappa_y} \quad \Rightarrow \quad \alpha_{\text{k}} = -76^{\circ}$$

The stresses for each point of the cross section can be computed with:

$$\sigma(y,z) = E \times (\varepsilon + \kappa_y \times y + \kappa_z \times z) \quad \text{N/mm}^2$$

The neutral axis *n.a.* can also be found with this latter expression by:

$$\varepsilon(y,z) = \varepsilon + \kappa_y \times y + \kappa_z \times z = 0 \quad \Leftrightarrow \quad \kappa_y \times y + \kappa_z \times z = 0 \quad \Leftrightarrow \quad y - 4z = 0$$

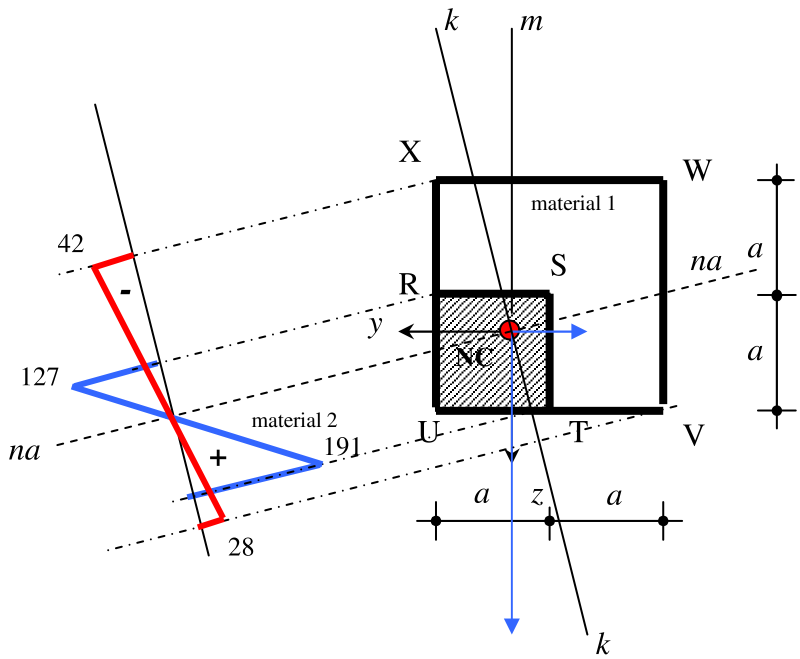
e) The stress distribution can be visualized with a few points. Only the four values marked in **bold** in the table on the next page were essential for the graphs.

Tabel : Stress in the specified points

Material	point	*y* [mm]	*z* [mm]	*E* [N/mm^2]	*Stress* [N/mm^2]
1	R	200	-100	200000	-14,1
	S	-100	-100	200000	-7,1
	T	-100	200	200000	21,2
	V	-400	200	200000	**28,3**
	W	-400	-400	200000	-28,3
	X	200	-400	200000	**-42,4**
2	R	200	-100	1800000	**-127,3**
	S	-100	-100	1800000	-63,7
	T	-100	200	1800000	**191,0**
	U	200	200	1800000	127,3

The neutral axis goes through the NC since the normal force *N* is zero. The red stress distribution represents the stresses in material 1 and the blue one represents material 2.

The moment *M* and thus the load *F* acts in the *x*-*m* plane. The curvature κ acts in the *x*-*k* plane.



f) The longitudinal force per unit length of beam in the interface between material 1 and 2 can be obtained with:

Material 2 (RSTU) is taken as the sliding element with cross sectional area (*a*):

$$s_x = -\frac{R_M^{(a)}}{M} \times V = -\frac{\frac{1}{4}(\sigma_R + \sigma_S + \sigma_T + \sigma_U) \times a^2}{M_z} \times V_z$$

$$s_x = -\frac{\frac{1}{4}(127,3) \times 300 \times 300}{1910 \times 10^6} \times 191 \times 10^3 = 286 \text{ N/mm}$$

g) The outer left kern point can be found by taking a neutral axis along VW. The location of the kernel point can be found with:

$$\begin{bmatrix} e_y \\ e_z \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} -1/(-400) \\ 0 \end{bmatrix} = \begin{bmatrix} 50 \\ 12,5 \end{bmatrix} \text{mm}$$

h) The principle direction of this cross section is at an angle of 45 degrees. For the principle coordinate system this cross section has one axis of symmetry. The maximum stiffness of this section then becomes:

$$EI_{1,2} = \tfrac{1}{2}(EI_{yy} + EI_{zz}) \pm \sqrt{\left(\tfrac{1}{2}(EI_{yy} - EI_{zz})\right)^2 + EI_{yz}^2}$$

$$EI_1 = 5400 \times 10^{12} \text{ Nmm}^2; \quad EI_2 = 3240 \times 10^{12} \text{ Nmm}^2$$

Maximum bending stiffness is therefore 5400×10^{12} Nmm^2.