TU Delft
Subfaculty of Civil Engineering
Structural Mechanics

Mark each page with your:
STUDENT NUMBER :
NAME :

Exam CT3109

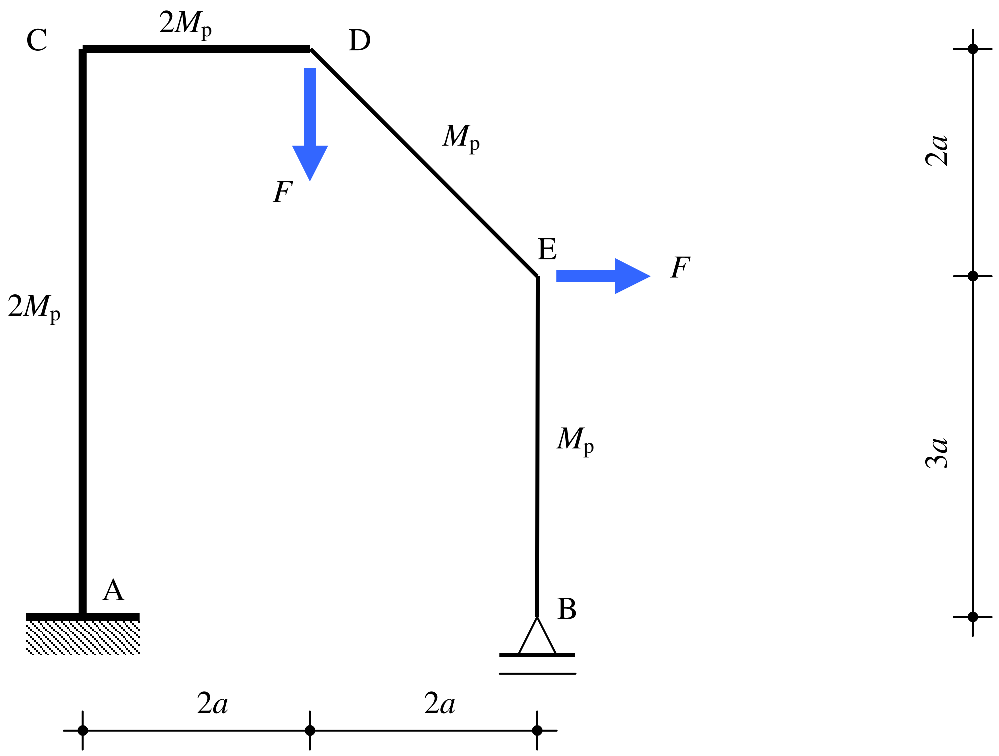
STRUCTURAL MECHANICS 4

11 april 2011,
09:00 – 12:00 hours

- This exam consists of 4 **problems**.
- Use for each problem a separate sheet of paper.
- Do not forget to mention your name and number on each paper.
- Work neat and tidy, the quality of the presentation can be used in the grading.
- The use of Phone's or computers, PDA's and /or Wifi or Blue Tooth equipment is not allowed. Turn off the equipment and remove it from your table.
- A scientific (programmable) calculator is allowed
- All required formulas can be found on the last pages of this exam
- Keep an eye on the clock and use the specified times per problem as guidance.

Problem 1 : Plasticity (45 min)

In the following figure a frame is shown which is fixed in A and supported in B with a horizontal roller. Concentrated forces act at D and E in the directions shown. Segment ACD has twice the strength of segment BED.



Questions:

a) Determine the possible collapse mechanisms and show these with small sketches.

b) Compute the collapse load F_p and prove the uniqueness of your solution.

c) Show the moment distribution at the moment of collapse.

Problem 2 : Influence lines (35 min)

The statically determinate hinged beam structure form the figure below consists of three beams which are connected though hinges at S1 and S2.

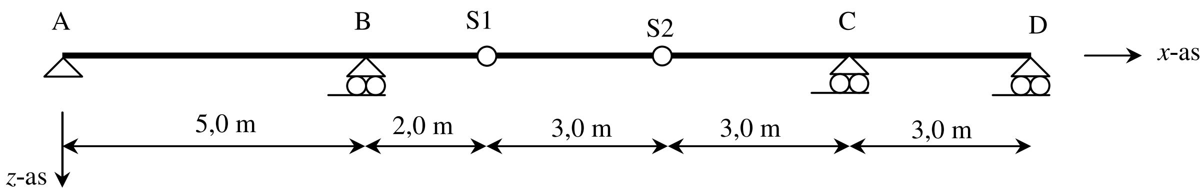


Figure 1 : Static determinate hinged beam

Questions:

a) Explain how an influence line for a force quantity can be determined.
b) Explain how an influence line for a displacement quantity can be determined.
c) Construct the influence line for the moment at B.
d) Construct the influence line for the moment at C.
e) Construct the influence line for the shear force directly to the left of C.
f) Construct the influence line for the shear force at mid span of span AB.
g) Construct the influence line for the for the support reaction at C.

h) Sketch the influence line for the displacement of hinge S2.

i) Mark the position of a constant distributed load on the structure in order to maximize the bending moment at one of the supports.

Problem 3 : work and energy methods (45 min)

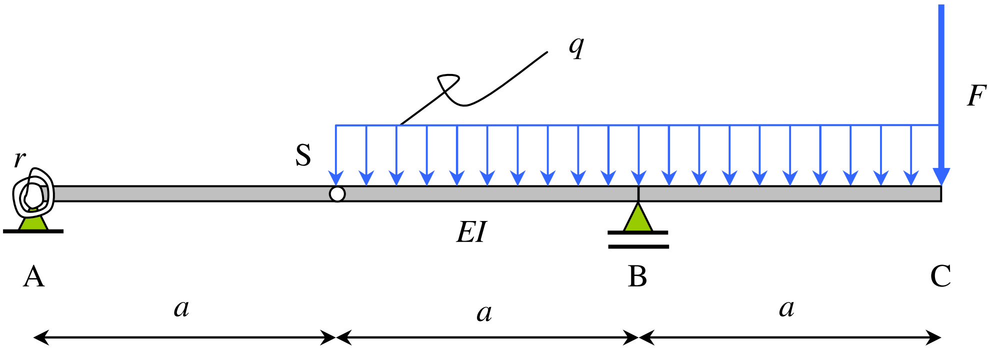
A hinged beam with a rotational spring attached at A is loaded by a concentrated load *F* and a distributed load *q*. S is a hinge connecting the two beam parts loaded in bending. The bending stiffness for the total structure is constant and denoted with *EI*.

The spring stiffness can be expressed in terms of the bending stiffness by using a dimensionless scalar:

$$\rho = \frac{r \cdot a}{EI} \quad \Rightarrow \quad r = \frac{\rho \cdot EI}{a}$$

The deformation energy stored in a rotational spring can be expressed as:

$$E_v = \frac{M_{veer}^2}{2r}$$



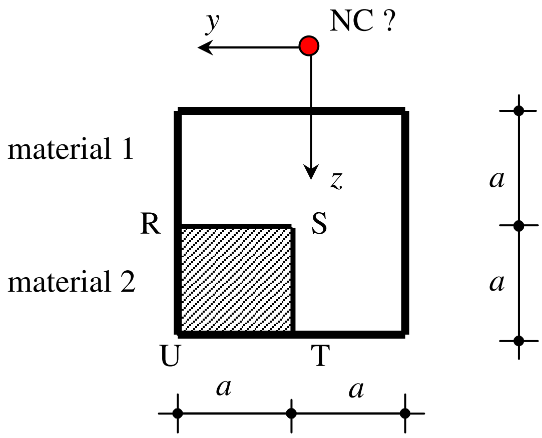
Questions:

a) Find the moment distribution fort his structure and draw this *Moment-line*.
b) Find the vertical displacement of the hinge S with a work or energy method. Write the displacement in terms of the given parameters and use the given scalar ρ for the stiffness of rotational spring.
c) Specify the amount of deformation energy stored in the structure due to the specified load and express this in terms of the parameters *F, q, a,* ρ en *EI.*
d) Find for : EI = 10000 kNm^2; $\rho = 2$, a = 4 m; q = 4 kN/m and F = 10 kN the deflection at S in mm and the maximum moment in kNm.
e) Find the amount of deformation energy for very small values of *r*.

HINT : use a sketch of the deformed structure!

Problem 4 (45 min)

An element subject to bending and shear is a composition of two materials. The geometry of the cross section is symmetrical and shown in the figure below. Both materials are perfectly bonded and the cross section can be regarded as an inhomogeneous cross section. The dimensions and Young's moduli of the materials can be found below. The beam axis coincides with the *x*-axis and the origin of the coordinate system is taken at the normal force center of the cross section. The cross section is loaded in the *x*-*z*-vlak with a shear force of 191 kN and a bending moment of 1910 kNm.



Given : $E_1 = 200\times10^3$ N/mm^2; $E_2 = 1800\times10^3$ N/mm^2; $a = 300$ mm

Questions:

a) Why is the origin of the coordinate system taken at the normal force center?
b) Find the location of the normal force center.
c) The constitutive matrix of the cross section is given below. Show the correctness of this matrix.

$$10^{12}\begin{bmatrix} 0,216 & 0 & 0 \\ 0 & 4320 & 1080 \\ 0 & 1080 & 4320 \end{bmatrix}$$ units N, mm

d) Find the position of the *neutral axis*, the plane of loading *m* and the plane of curvature *k* and draw these lines in one graph.
e) Show in a second graph the stress distribution of the cross section for both materials.

f) Find the total longitudinal force in the interface RST between material 1 and 2 per unit of length.

g) Find the most outer left point of the kernel of this cross section and mark this point in one of the graphs of the cross section.

h) In which plane will this cross section have the maximum bending stiffness and what is the magnitude (value) of this stiffness?

FORMULAS

Inhomogeous and/or unsymmetrical cross sections :

$$\varepsilon(y,z) = \varepsilon + \kappa_y y + \kappa_z z \qquad \text{and:} \quad \sigma(y,z) = E(y,z) \times \varepsilon(y,z)$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e_y \\ e_z \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} -1/y_1 \\ -1/z_1 \end{bmatrix}$$

$$s_x^{(a)} = -\frac{V_y ES_y^{(a)}}{EI_{yy}} - \frac{V_z ES_z^{(a)}}{EI_{zz}}; \quad \text{or:} \quad s_x^{(a)} = -\frac{R_M^{(a)}}{M} V; \quad \sigma_{xt} = \frac{s_x^{(a)}}{b^{(a)}}$$

$$\tan(2\alpha) = \frac{2EI_{yz}}{\left(EI_{yy} - EI_{zz}\right)}; \quad EI_{1,2} = \tfrac{1}{2}\left(EI_{yy} + EI_{zz}\right) \pm \sqrt{\left(\tfrac{1}{2}\left(EI_{yy} - EI_{zz}\right)\right)^2 + EI_{yz}^2}$$

Deformation energy:

$$E_v = \int \tfrac{1}{2} EA\varepsilon^2 \mathrm{d}x \quad \text{(extension)}$$

$$E_v = \int \tfrac{1}{2} EI\kappa^2 \mathrm{d}x \quad \text{(bending)}$$

Complementairy energy:

$$E_c = \int \frac{N^2}{2EA} \mathrm{d}x \quad \text{(extensie)}$$

$$E_c = \int \frac{M^2}{2EI} \mathrm{d}x \quad \text{(buiging)}$$

Castigliano's theorema's:

$$F_i = \frac{\partial E_v}{\partial u_i} \qquad u_i = \frac{\partial E_c}{\partial F_i}$$

Kinematic relations:

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\kappa = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

Constitutive relations:

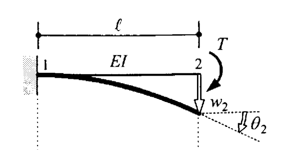
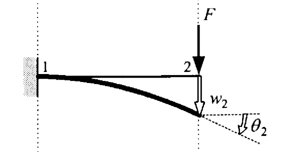
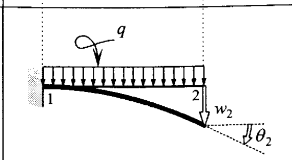
$$N = EA.\varepsilon$$

$$M = EI.\kappa$$

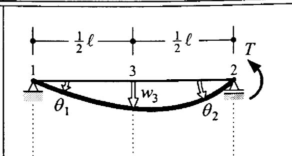
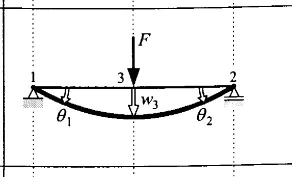
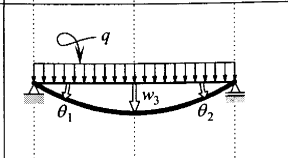
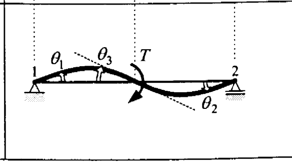
Work method with unit load:

$$u = \int \frac{M(x)m(x)dx}{EI}$$

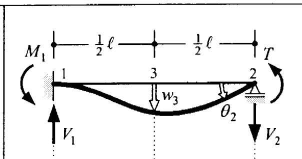
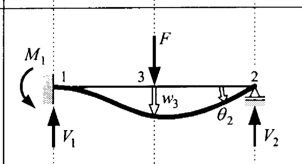
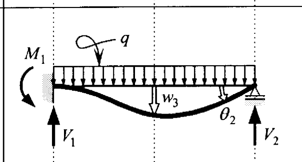
vergeet-mij-nietjes

(1)		$\theta_2 = \frac{T\ell}{EI}$; $w_2 = \frac{T\ell^2}{2EI}$
(2)		$\theta_2 = \frac{F\ell^2}{2EI}$; $w_2 = \frac{F\ell^3}{3EI}$
(3)		$\theta_2 = \frac{q\ell^3}{6EI}$; $w_2 = \frac{q\ell^4}{8EI}$

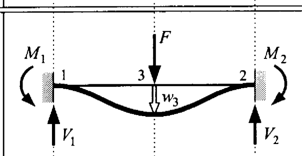
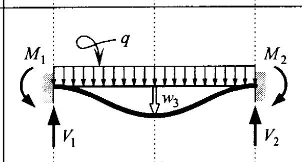
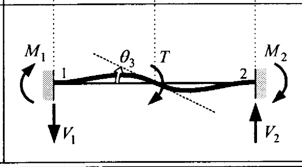
vrij opgelegde ligger (statisch bepaald)

(4)		$\theta_1 = \frac{1}{6}\frac{T\ell}{EI}$; $\theta_2 = \frac{1}{3}\frac{T\ell}{EI}$; $w_3 = \frac{1}{16}\frac{T\ell^2}{EI}$
(5)		$\theta_1 = \theta_2 = \frac{1}{16}\frac{F\ell^2}{EI}$; $w_3 = \frac{1}{48}\frac{F\ell^3}{EI}$
(6)		$\theta_1 = \theta_2 = \frac{1}{24}\frac{q\ell^3}{EI}$; $w_3 = \frac{5}{384}\frac{q\ell^4}{EI}$
(a)		$\theta_1 = \theta_2 = \frac{1}{24}\frac{T\ell}{EI}$; $\theta_3 = \frac{1}{12}\frac{T\ell}{EI}$; $w_3 = 0$

statisch onbepaalde ligger (enkelzijdig ingeklemd)

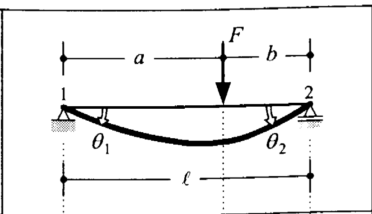
(7)		$\theta_2 = \frac{1}{4}\frac{T\ell}{EI}$; $w_3 = \frac{1}{32}\frac{T\ell^2}{EI}$ $M_1 = \frac{1}{2}T$; $V_1 = V_2 = \frac{3}{2}\frac{T}{\ell}$
(8)		$\theta_2 = \frac{1}{32}\frac{F\ell^2}{EI}$; $w_3 = \frac{7}{768}\frac{F\ell^3}{EI}$ $M_1 = \frac{3}{16}F\ell$; $V_1 = \frac{11}{16}F$; $V_2 = \frac{5}{16}F$
(9)		$\theta_2 = \frac{1}{48}\frac{q\ell^3}{EI}$; $w_3 = \frac{1}{192}\frac{q\ell^4}{EI}$ $M_1 = \frac{1}{8}q\ell^2$; $V_1 = \frac{5}{8}q\ell$; $V_2 = \frac{3}{8}q\ell$

statisch onbepaalde ligger (tweezijdig ingeklemd)

(10)		$w_3 = \frac{1}{192}\frac{F\ell^3}{EI}$ $M_1 = M_2 = \frac{1}{8}F\ell$; $V_1 = V_2 = \frac{1}{2}F$
(11)		$w_3 = \frac{1}{384}\frac{q\ell^4}{EI}$ $M_1 = M_2 = \frac{1}{12}q\ell^2$; $V_1 = V_2 = \frac{1}{2}q\ell$
(b)		$\theta_3 = \frac{1}{16}\frac{T\ell}{EI}$; $w_3 = 0$ $M_1 = M_2 = \frac{1}{4}T$; $V_1 = V_2 = \frac{3}{2}\frac{T}{\ell}$

Enkele formules voor prismatische liggers met buigstijfheid EI.
T, F en q zijn belastingen door resp. een koppel, kracht en gelijkmatig verdeelde belasting.
M_i en V_i zijn het buigend moment en de dwarskracht op einddoorsnede i van de ligger ten gevolge van de oplegreacties.

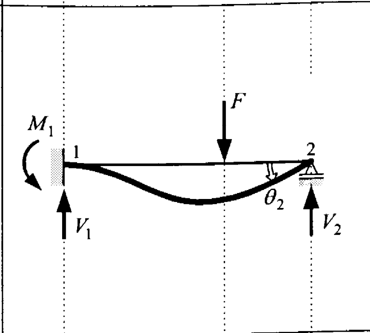
(c)



$$\theta_1 = \frac{Fab(\ell+b)}{6EI\ell} = \frac{F\ell^2}{6EI}\left(2\frac{a}{\ell} - 3\frac{a^2}{\ell^2} + \frac{a^3}{\ell^3}\right)$$

$$\theta_2 = \frac{Fab(\ell+a)}{6EI\ell} = \frac{F\ell^2}{6EI}\left(\frac{a}{\ell} - \frac{a^3}{\ell^3}\right)$$

(d)



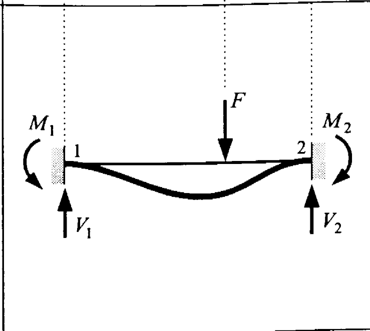
$$M_1 = \frac{Fb(\ell^2-b^2)}{2\ell^2} = F\ell\left(\frac{a}{\ell} - \frac{3}{2}\frac{a^2}{\ell^2} + \frac{1}{2}\frac{a^3}{\ell^3}\right)$$

$$V_1 = \frac{Fb(3\ell^2-b^2)}{2\ell^3} = F\left(1 - \frac{3}{2}\frac{a^2}{\ell^2} + \frac{1}{2}\frac{a^3}{\ell^3}\right)$$

$$V_2 = \frac{Fa^2(3\ell-a)}{2\ell^3} = F\left(\frac{3}{2}\frac{a^2}{\ell^2} - \frac{1}{2}\frac{a^3}{\ell^3}\right)$$

$$\theta_2 = \frac{Fa^2b}{4EI\ell} = \frac{F\ell^2}{4EI}\left(\frac{a^2}{\ell^2} - \frac{a^3}{\ell^3}\right)$$

(e)



$$M_1 = \frac{Fab^2}{\ell^2} = F\ell\left(\frac{a}{\ell} - 2\frac{a^2}{\ell^2} + \frac{a^3}{\ell^3}\right)$$

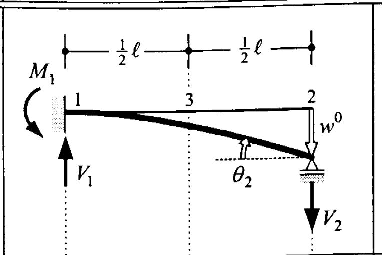
$$V_1 = \frac{Fb^2(\ell+2a)}{\ell^3} = F\left(1 - 3\frac{a^2}{\ell^2} + 2\frac{a^3}{\ell^3}\right)$$

$$M_2 = \frac{Fa^2b}{\ell^2} = F\ell\left(\frac{a^2}{\ell^2} - \frac{a^3}{\ell^3}\right)$$

$$V_2 = \frac{Fa^2(\ell+2b)}{\ell^3} = F\ell\left(3\frac{a^2}{\ell^2} - 2\frac{a^3}{\ell^3}\right)$$

drie bij-de handjes

(f)

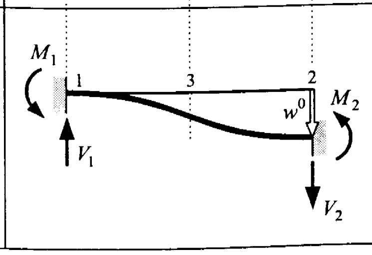


$$M_1 = \frac{3EI}{\ell^2}w^0; \quad V_1 = V_2 = \frac{3EI}{\ell^3}w^0$$

$$\theta_2 = \frac{3}{2}\frac{w^0}{\ell}$$

$$\theta_3 = \frac{9}{8}\frac{w^0}{\ell}; \quad w_3 = \frac{5}{16}w^0$$

(g)



$$M_1 = M_2 = \frac{6EI}{\ell^2}w^0; \quad V_1 = V_2 = \frac{12EI}{\ell^3}w^0$$

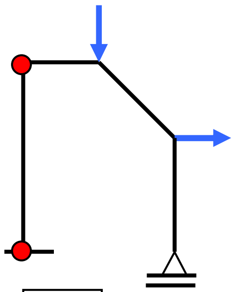
$$\theta_3 = \frac{3}{2}\frac{w^0}{\ell}; \quad w_3 = \frac{1}{2}w^0$$

zettingen

ANSWERS

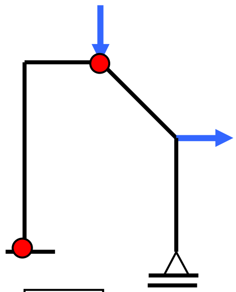
Problem 1 : Plasticity

a) The given structure is statically indeterminate with a degree of one. To obtain a mechanism, two additional hinges are required. Hinges can occur at four positions. Thus, six mechanisms have to be investigated. However, moments can not occur at E due to the loading and support conditions at B thus only three mechanisms are left to be considered. Possible mechanisms are shown below. Each mechanism is shown in detail on the following pages.



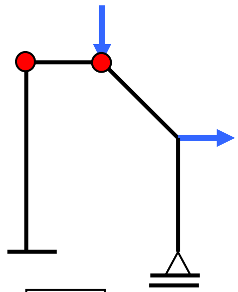
1

$$F_p = \frac{4M_p}{5a}$$



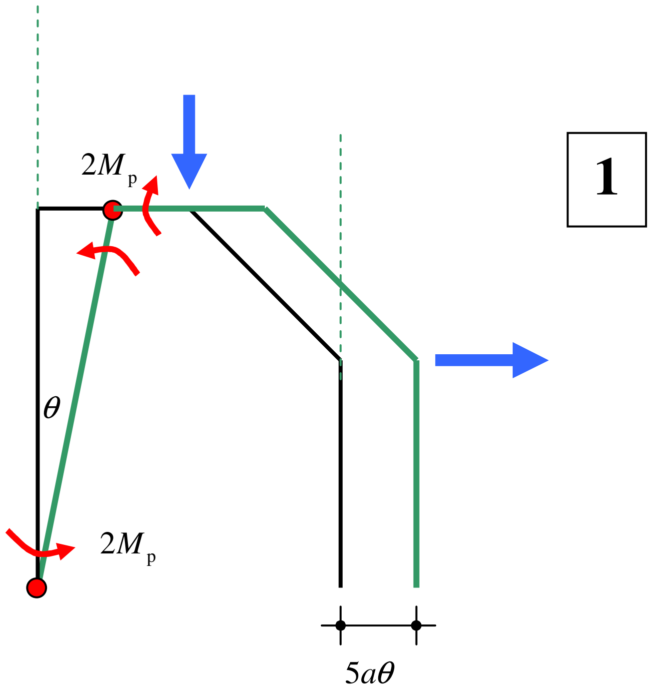
2

$$F_p = \frac{4M_p}{9a}$$



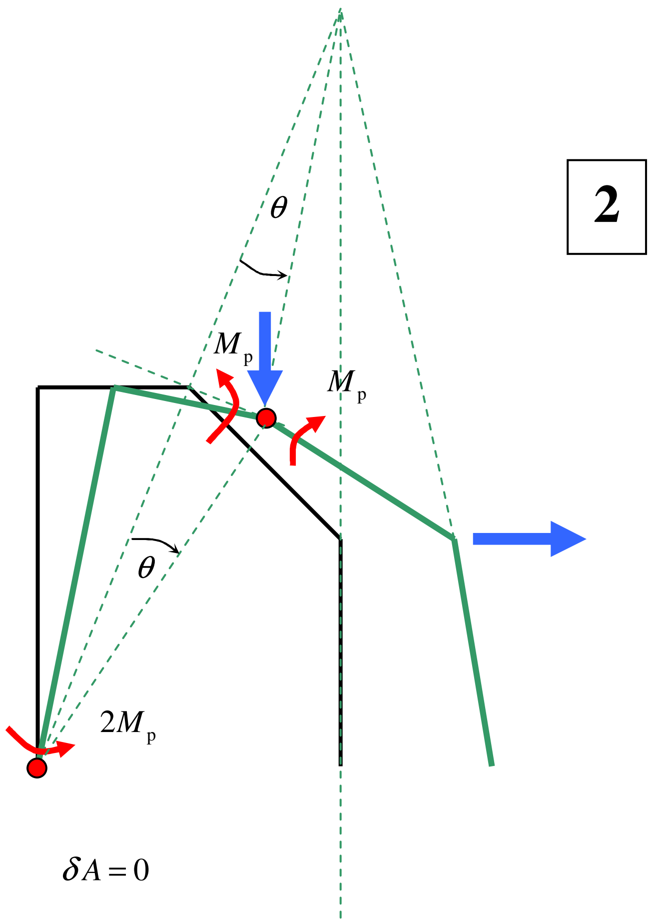
3

$$F_p = \frac{M_p}{a}$$



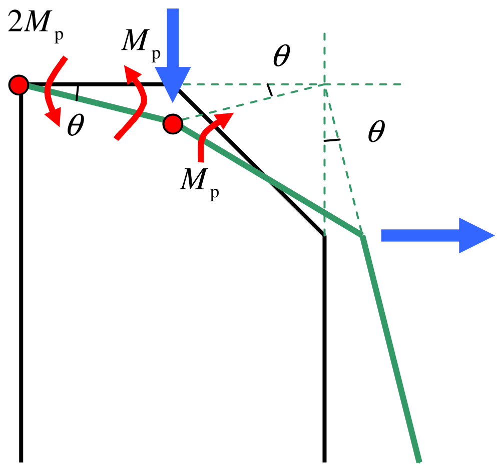
$$\delta A = 0$$

$$-2M_p \times \delta\theta - 2M_p \times \delta\theta - 2M_p \times 0(!) + F_p \times 5a\delta\theta = 0 \quad \Leftrightarrow \quad F_p = \frac{4M_p}{5a}$$



$$\delta A = 0$$

$$-2M_p \times \delta\theta - M_p \times \delta\theta - M_p \times \delta\theta + F_p \times 2a\delta\theta + F_p \times 7a\delta\theta = 0 \quad \Leftrightarrow \quad F_p = \frac{4M_p}{9a}$$



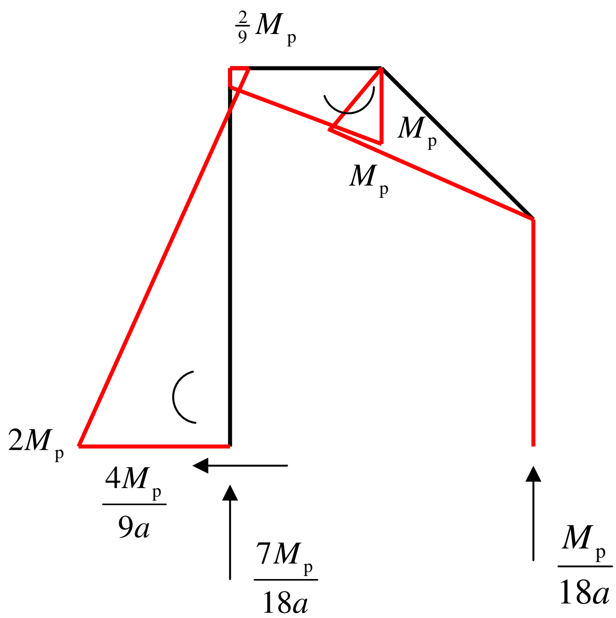
3

$$\delta A = 0$$

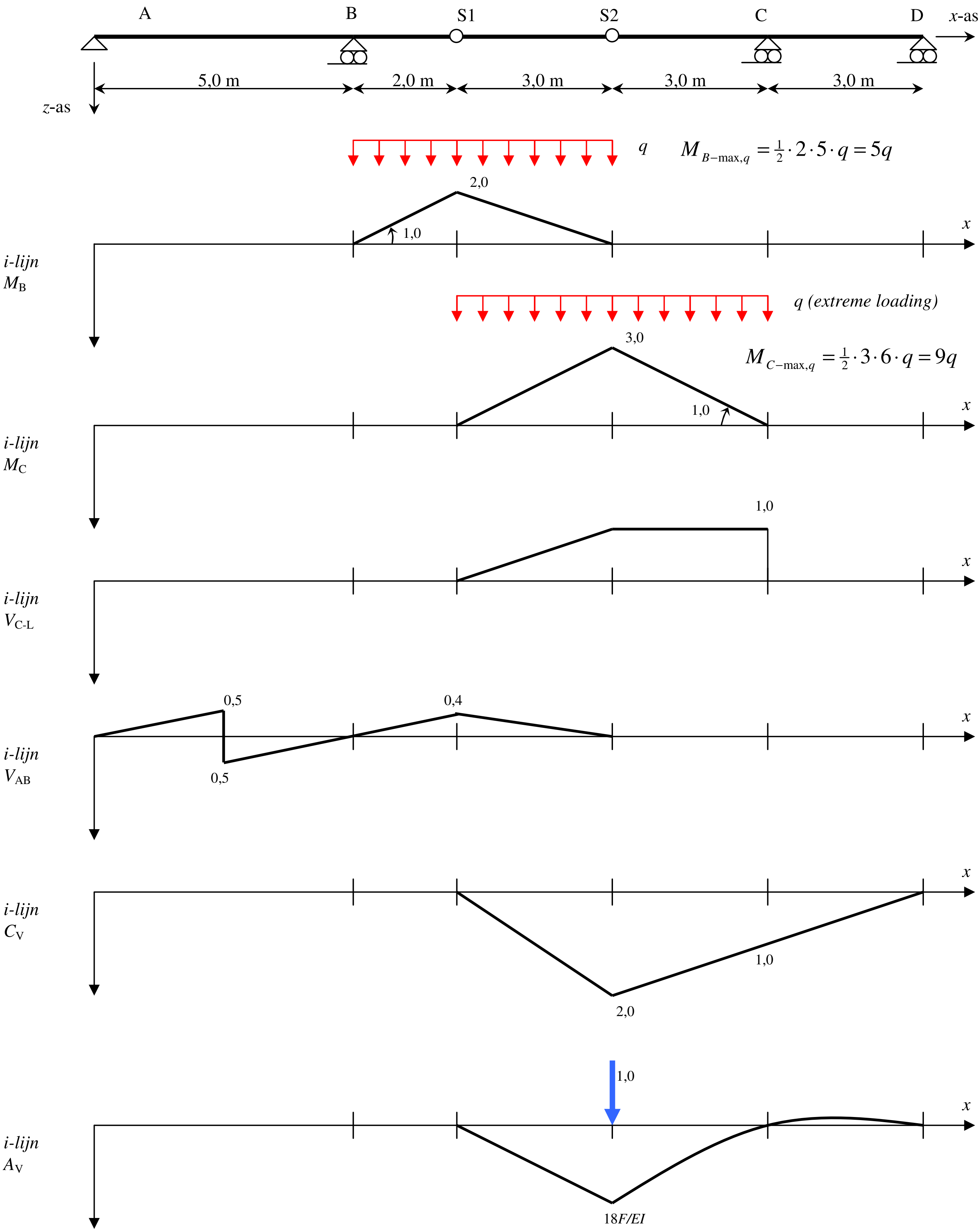
$$-2M_p \times \delta\theta - M_p \times \delta\theta - M_p \times \delta\theta + F_p \times 2a\delta\theta + F_p \times 2a\delta\theta = 0 \quad \Leftrightarrow F_p = \frac{M_p}{a}$$

a) The ultimate load is the lowest load found: $F_p = \dfrac{4M_p}{9a} = 0,444\dfrac{M_p}{a}$

b) To find the correct moment distribution start with the support reactions. The moment distribution shows a statically admissible distribution in which at no cross section the bending moments exceeds the utltimate plastic capacity of the cross section. (**uniqueness theorem**)

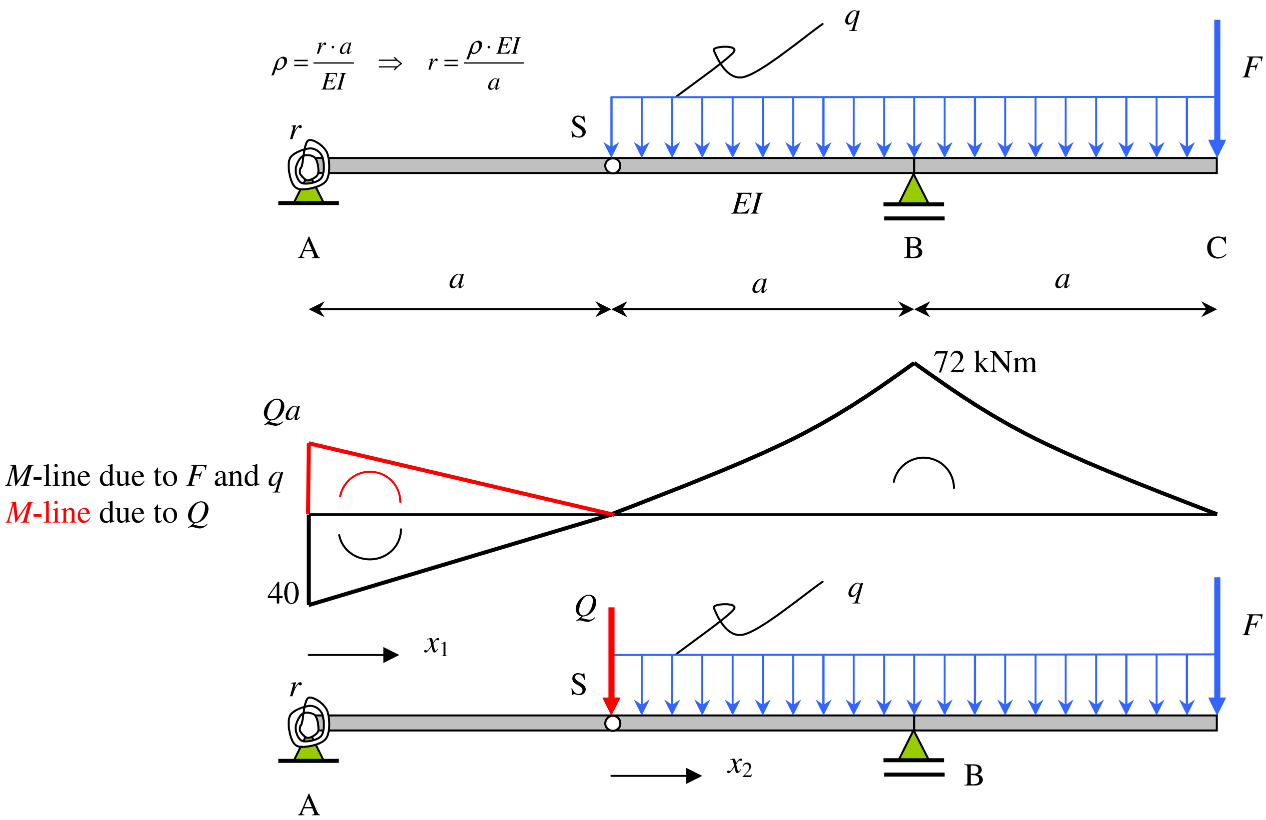


Problem 2 : Influence lines



Problem 3 : Work and Energy Methods

a) The moment distribution can be found base don equilibrium. The distributed load q is in fact balanced thus only the concentrated load is contributing to the moment in the spring and the member AS. (first years knowledge !!)



b) To find the displacement at S a dummy load Q is required. In red the moment distribution due to the dummy load is presented. Since the moment distribution over SB and BC are exhibiting mirror symmetry, only part SB has to be examined. Using a local coordinate for part AS and SB denoted with x_1 and x_2, the deformation energy can easily be expressed in terms of F, q and Q :

$$E_v = \frac{M_v^2}{2r} + \int_{x_1=0}^{x_1=a} \frac{\left((F-Q)(a-x_1)\right)^2}{2EI} dx_1 + 2\int_{x_2=0}^{x_2=a} \frac{\left(Fx_2 + \frac{1}{2}qx_2^2\right)^2}{2EI} dx_2 \qquad \text{met:}$$

$$M_v = (F-Q) \cdot a$$

Using Castigliano's theorema solves the deflection at S: (note: 2nd integral is irrelevant)

$$w_S = \frac{\partial E_v}{\partial Q} = -\frac{(F-Q)a^3}{\rho EI} - \frac{(F-Q)a^3}{3EI}$$

The dummy load is zero which results in: $w_S = -\frac{Fa^3(3+\rho)}{3\rho EI} = -\frac{Fa^3}{\rho EI} - \frac{Fa^3}{3EI}$

Hinge S will move upwards due tot the load F.

c) Using the given values results in a displacement of 0,0533 m upwards. This displacement is independent of the distributed load q!

d) The total deformation energy stored due to the loading is: (note : 2nd integral is essential)

$$E_v = \frac{M_v^2}{2r} + \int_{x_1=0}^{x_1=a} \frac{\left(F\left(a-x_1\right)\right)^2}{2EI} dx_1 + 2\int_{x_2=0}^{x_2=a} \frac{\left(Fx_2 + \frac{1}{2}qx_2^2\right)^2}{2EI} dx_2$$

$$E_v = \frac{a^3\left(10F^2(1+\rho) + qa\rho(5F + qa\rho)\right)}{20\rho EI}$$

e) If the spring stiffness becomes very small, point A turns into a hinge. A mechanism will occur. The load can move infinitely and thus produce infinite work. The bending deformation will be small compared to the deformation energy stored in the spring. So practically all deformation energy has to be taken by the spring(!) which results in hardly any deformation energy (and curvature) in the elements loaded in bending. The bars will remain straight:

$$\underset{\lim \rho \to 0}{E_v} = F \cdot w = \frac{\left(F \cdot a\right)^2}{2r} = \infty$$

Problem 4 : Non-symmetrical cross sections

a) See the lecture notes.
b) The axial stiffness of the cross section can be found with:

$$EA = 2a \times 2a \times E_1 - a \times a \times E_1 + a \times a \times E_2 = 216 \times 10^9$$

The origin of the coordinate system used is located at the NC. The vertical position of the NC with respect to the upper side of the cross section is:

$$\Delta z_{NC} = 400 \ \text{mm}$$

The horizontal position with respect to the left side of the cross section is:

$$\Delta y_{NC} = 200 \ \text{mm}$$

c) The *cross sectional constitutive relation* relates the sectional forces to the deformations if the cross section. The bending stiffnesses can be found using the strategy outlined in the lecure notes. This example is very basic so only answers are presented here:

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix} \text{cross sectional constitutive relation [N, mm]}$$

$$K = 10^{12} \begin{bmatrix} 0,216 & 0 & 0 \\ 0 & 4320 & 1080 \\ 0 & 1080 & 4320 \end{bmatrix} \quad \overline{f} = \begin{bmatrix} 0 \\ 0 \\ 1910 \times 10^6 \end{bmatrix}$$

d) Since this structure is loaded in bending only, the strain ε at the NC must be zero. The curvatures can be found with the constitutive relation:

$$\varepsilon = \frac{N}{EA} = 0$$

$$\kappa_y = \frac{1}{EI_{yy}EI_{zz} - EI_{yz}^2}\left(EI_{zz} \times M_y - EI_{yz} \times M_z \right) = -0,1179 \times 10^{-6}$$

$$\kappa_z = \frac{1}{EI_{yy}EI_{zz} - EI_{yz}^2}\left(-EI_{yz} \times M_y + EI_{yy} \times M_z \right) = 0,4716 \times 10^{-6}$$

The direction of the *plane of loading* and the *plane of curvature* can be obtained with:

$$\tan \alpha_m = \frac{M_z}{M_y} \quad \Rightarrow \quad \alpha_{\text{m}} = 90^{\circ}; \qquad \tan \alpha_k = \frac{\kappa_z}{\kappa_y} \quad \Rightarrow \quad \alpha_{\text{k}} = -76^{\circ}$$

The stresses for each point of the cross section can be computed with:

$$\sigma(y,z) = E \times (\varepsilon + \kappa_y \times y + \kappa_z \times z) \quad \text{N/mm}^2$$

The neutral axis *n.a.* can also be found with this latter expression by:

$$\varepsilon(y,z) = \varepsilon + \kappa_y \times y + \kappa_z \times z = 0 \quad \Leftrightarrow \quad \kappa_y \times y + \kappa_z \times z = 0 \quad \Leftrightarrow \quad y - 4z = 0$$

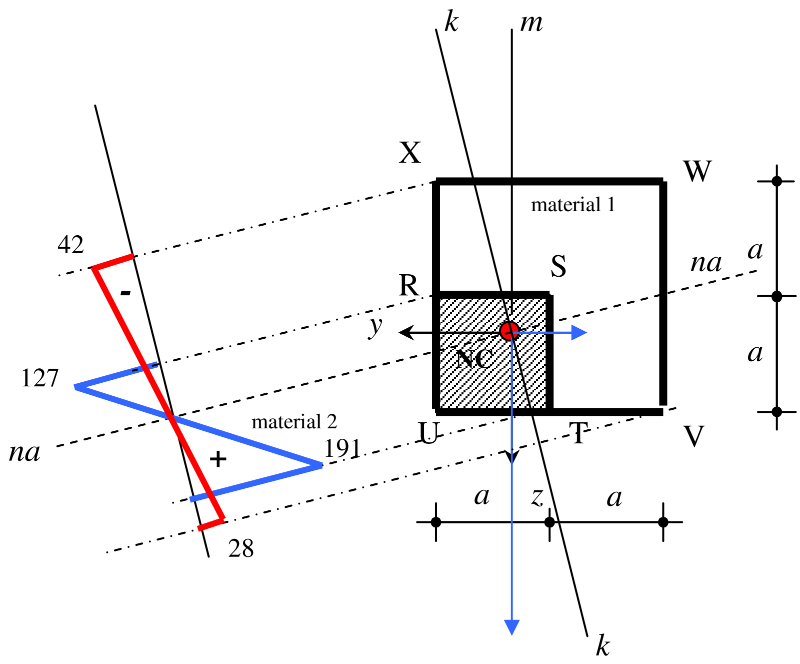
e) The stress distribution can be visualized with a few points. Only the four values marked in **bold** in the table on the next page were essential for the graphs.

Tabel : Stress in the specified points

Material	point	*y* [mm]	*z* [mm]	*E* [N/mm^2]	*Stress* [N/mm^2]
1	R	200	-100	200000	-14,1
	S	-100	-100	200000	-7,1
	T	-100	200	200000	21,2
	V	-400	200	200000	**28,3**
	W	-400	-400	200000	-28,3
	X	200	-400	200000	**-42,4**
2	R	200	-100	1800000	**-127,3**
	S	-100	-100	1800000	-63,7
	T	-100	200	1800000	**191,0**
	U	200	200	1800000	127,3

The neutral axis goes through the NC since the normal force *N* is zero. The red stress distribution represents the stresses in material 1 and the blue one represents material 2.

The moment *M* and thus the load *F* acts in the *x*-*m* plane. The curvature κ acts in the *x*-*k* plane.



f) The longitudinal force per unit length of beam in the interface between material 1 and 2 can be obtained with:

Material 2 (RSTU) is taken as the sliding element with cross sectional area (*a*):

$$s_x = -\frac{R_M^{(a)}}{M} \times V = -\frac{\frac{1}{4}(\sigma_R + \sigma_S + \sigma_T + \sigma_U) \times a^2}{M_z} \times V_z$$

$$s_x = -\frac{\frac{1}{4}(127,3) \times 300 \times 300}{1910 \times 10^6} \times 191 \times 10^3 = 286 \text{ N/mm}$$

g) The outer left kern point can be found by taking a neutral axis along VW. The location of the kernel point can be found with:

$$\begin{bmatrix} e_y \\ e_z \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} -1/(-400) \\ 0 \end{bmatrix} = \begin{bmatrix} 50 \\ 12,5 \end{bmatrix} \text{mm}$$

h) The principle direction of this cross section is at an angle of 45 degrees. For the principle coordinate system this cross section has one axis of symmetry. The maximum stiffness of this section then becomes:

$$EI_{1,2} = \tfrac{1}{2}(EI_{yy} + EI_{zz}) \pm \sqrt{\left(\tfrac{1}{2}(EI_{yy} - EI_{zz})\right)^2 + EI_{yz}^2}$$

$$EI_1 = 5400 \times 10^{12} \text{ Nmm}^2; \quad EI_2 = 3240 \times 10^{12} \text{ Nmm}^2$$

Maximum bending stiffness is therefore 5400×10^{12} Nmm^2.