Subfaculty of Civil Engineering
Structural Mechanics

Mark each page with your:
STUDENT NUMBER :
NAME :

Exam CIE3109

STRUCTURAL MECHANICS 4

14 April 2014,
09:00 – 12:00 hours

- This exam consists of 4 **problems**.
- Use for each problem a separate sheet of paper.
- Do not forget to mention your name and number on each paper.
- Work neat and tidy, the quality of the presentation can be used in the grading.
- The use of Phone's or computers, PDA's and /or Wifi or Blue Tooth equipment is not allowed. Turn off the equipment and remove it from your table.
- A scientific (programmable) calculator is allowed
- All required formulas can be found on the last pages of this exam
- Keep an eye on the clock and use the specified times per problem as guidance.

Problem 1 : Influence lines (45 min)

A load system which consists of three concentrated loads of 15 kN each moves over a hinged beam as shown in figure 1. The hinged beam has hinges S1 and S2.

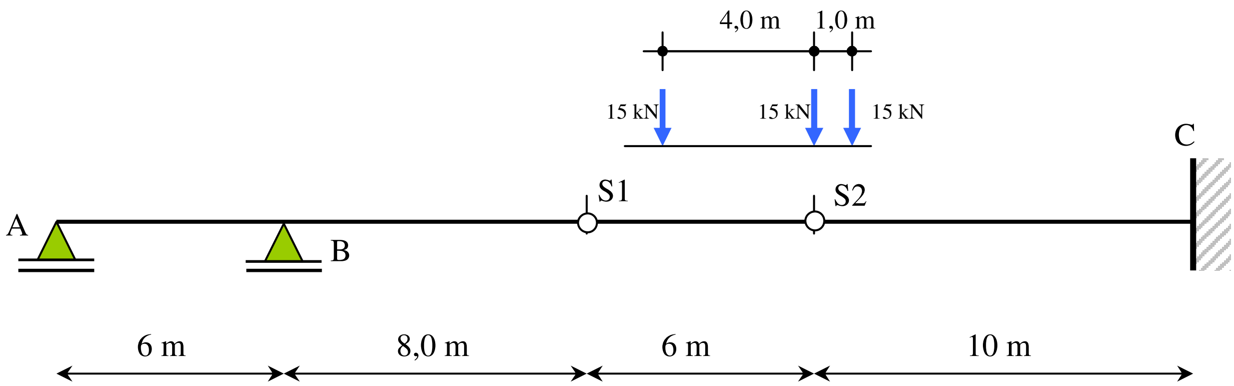


Figure 1 : Hinged beam

Questions:

a) Construct the influence line for the support reaction at A,
b) Construct the influence line for the moment at C,
c) Construct the influence line for the shear force directly to the left of S1,
d) Construct the influence line for the shear force directly to the right of S2,
e) Construct the influence line for the shear force directly to the right of support B,
f) Construct the influence line for the deflection of hinge S2

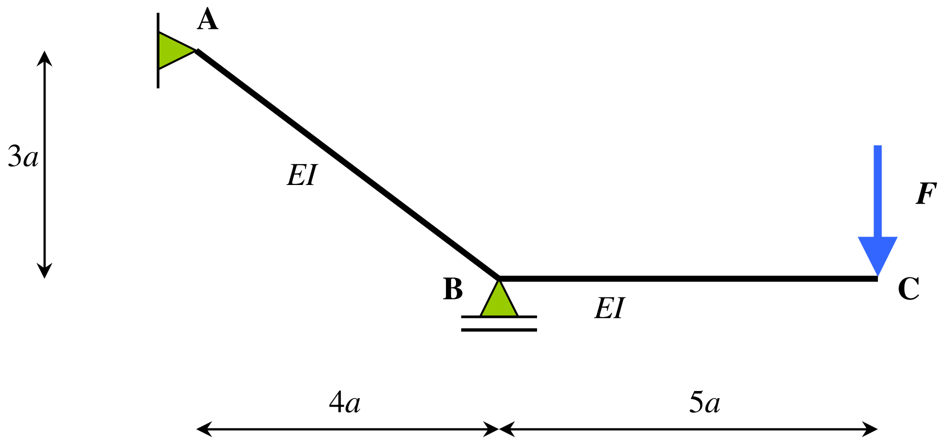
g) Sketch the influence line for the rotation at B
h) Sketch the influence line for the deflection at midspan between B and hinge S1,

i) Find the most unfavorable position of the load system for the moment at C,
j) Find the most unfavorable position of the load system for the shear force at B.

Note : "Construct" requires a correct sketch and the computed values of the influence factors at key points in the graph. Thus showing a qualitative and quantitative result. Sketching requires only a qualitative result from which it should become clear if member parts will remain straight or become curved.

Problem 2 : work and energy methods (40 min)

A bent beam is loaded with a concentrated load at C. The prismatic beam has a bending stiffness *EI* and axial deformation can be neglected.



Questions:

a) Sketch the moment distribution for this load case. Show clearly the deformation signs and write down the extreme value of the bending moment.

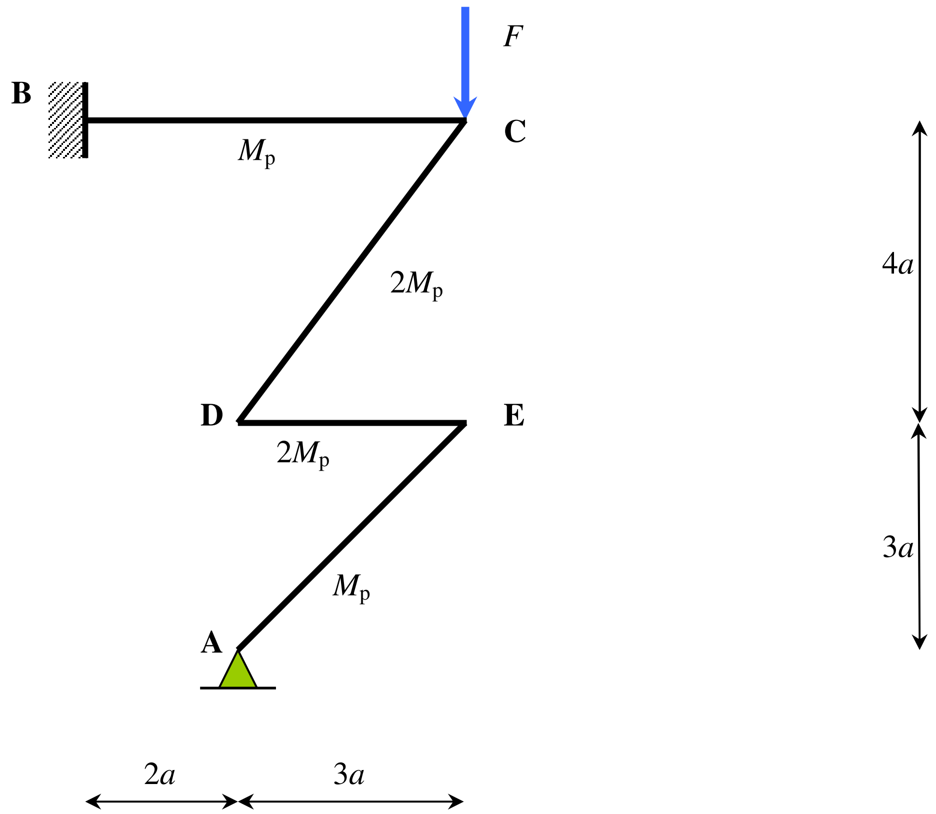
b) Find the expression for the total amount of strain energy due to bending expressed in *F*, *a* and *EI*.

c) Find, using Castigliano's theoreme, the deflection at C expressed in *F*, *a* and *EI*.

d) Find, , using Castigliano's theoreme, the rotation at B expressed in *F*, *a* and *EI*.

Problem 3 : Plasticity (45 min)

In the following figure a frame is shown. The concentrated load acts at C and BC and AE have a strength of M_p. The parts DC and DE have a strength $2M_p$.



Questions:

a) Determine the possible collapse mechanisms and show these with small sketches.

b) Compute the collapse load F_p and prove the uniqueness of your solution.

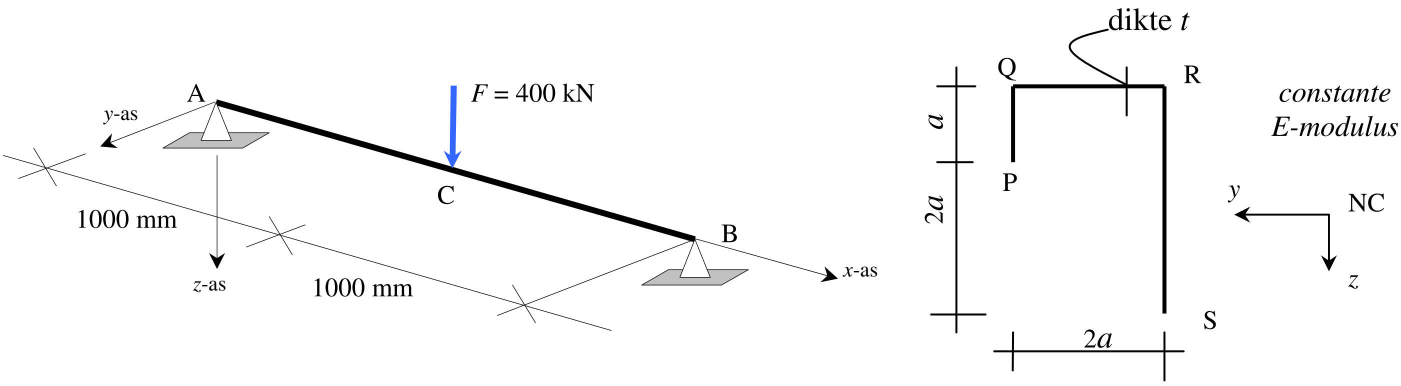
c) Show the moment distribution at collapse.

d) What does the uniqueness theorem of Prager mean?

Problem 4 **(50 min)**

A simply supported beam with an asymmetrical cross section is loaded in the *z*-direction with a concentrated load F at mid span as indicated in figure (a).

The cross section consists of a so-called thin walled section with constant thickness *t* as shown in figure (b). We assume that the load acts at the shear centre of the cross section. Therefore no torsion is generated. The influence of the deformation due to a possible normal force is neglected.



(a) Constructie met belasting (b) Doorsnede

Gegeven : $E = 2\times10^5$ N/mm^2; a = 180 mm; t = 10 mm;

Questions :

a) Find the location of the normal force centre en prove that for the given cross section at C holds:

$$EI_{yy} = 4ta^3E; \quad EI_{yz} = -\frac{7}{3}ta^3E; \quad EI_{zz} = \frac{31}{6}ta^3E$$

b) Draw the cross section (scale 1:5) and show the position of the neutral axis, the plane of loading *m-m* and the plane of curvature *k-k*.

c) Show that the neutral axis can be represented by:

$$70y + 120z = 0$$

d) Determine the stress distribution for the cross section at C. Show in a table the numerical values of the stresses at P, Q, R and S.

Answer with respect to the shear stress distribution the following questions

e) The shape and sign of the shear stress distribution for RS,

f) Find the location and the magnitude of the maximum shear stress between RS,

g) Draw the shear stress distribution for RS and show the direction and its value at characteristic points.

FORMULAS

Inhomogeous and/or asymmetrical cross sections :

$$\varepsilon(y,z) = \varepsilon + \kappa_y y + \kappa_z z \qquad \text{and:} \quad \sigma(y,z) = E(y,z) \times \varepsilon(y,z)$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e_y \\ e_z \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} -1/y_1 \\ -1/z_1 \end{bmatrix}$$

$$s_x^{(a)} = -\frac{V_y ES_y^{(a)}}{EI_{yy}} - \frac{V_z ES_z^{(a)}}{EI_{zz}}; \quad \text{or:} \quad s_x^{(a)} = -\frac{R_M^{(a)}}{M} V; \quad \sigma_{xt} = \frac{s_x^{(a)}}{b^{(a)}}$$

$$\tan(2\alpha) = \frac{2EI_{yz}}{\left(EI_{yy} - EI_{zz}\right)}; \quad EI_{1,2} = \tfrac{1}{2}\left(EI_{yy} + EI_{zz}\right) \pm \sqrt{\left(\tfrac{1}{2}\left(EI_{yy} - EI_{zz}\right)\right)^2 + EI_{yz}^2}$$

Deformation energy:

$$E_v = \int \tfrac{1}{2} EA\varepsilon^2 \mathrm{d}x \quad \text{(extension)}$$

$$E_v = \int \tfrac{1}{2} EI\kappa^2 \mathrm{d}x \quad \text{(bending)}$$

Kinematic relations:

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\kappa = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

Complementary energy:

$$E_c = \int \frac{N^2}{2EA} \mathrm{d}x \quad \text{(extension)}$$

$$E_c = \int \frac{M^2}{2EI} \mathrm{d}x \quad \text{(bending)}$$

Constitutive relations:

$$N = EA.\varepsilon$$

$$M = EI.\kappa$$

Castigliano's theorema's:

$$F_i = \frac{\partial E_v}{\partial u_i} \qquad u_i = \frac{\partial E_c}{\partial F_i}$$

Work method with unit load:

$$u = \int \frac{M(x)m(x)dx}{EI}$$

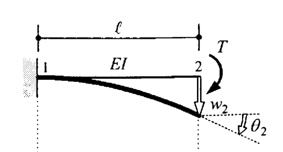
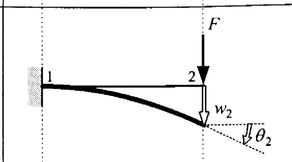
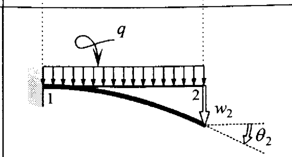
Rayleigh:

$$F_k = \frac{E_v}{\int \tfrac{1}{2} \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x}$$

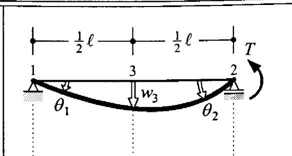
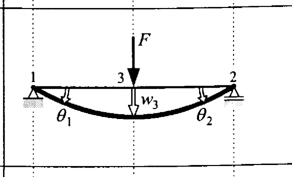
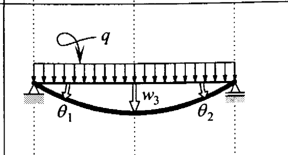
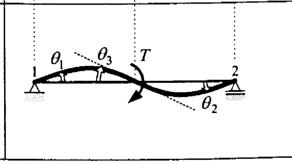
Math tools:

none

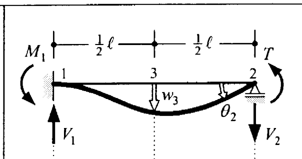
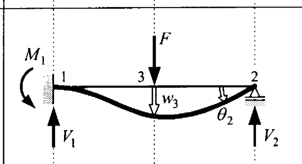
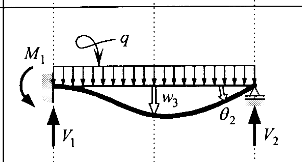
vergeet-mij-nietjes

(1)		$\theta_2 = \frac{T\ell}{EI}$; $w_2 = \frac{T\ell^2}{2EI}$
(2)		$\theta_2 = \frac{F\ell^2}{2EI}$; $w_2 = \frac{F\ell^3}{3EI}$
(3)		$\theta_2 = \frac{q\ell^3}{6EI}$; $w_2 = \frac{q\ell^4}{8EI}$

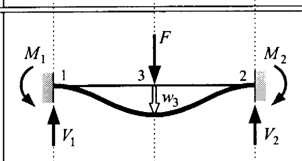
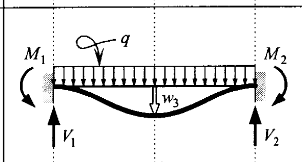
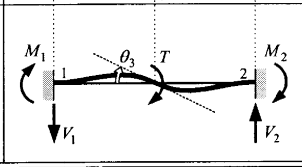
vrij opgelegde ligger (statisch bepaald)

(4)		$\theta_1 = \frac{1}{6}\frac{T\ell}{EI}$; $\theta_2 = \frac{1}{3}\frac{T\ell}{EI}$; $w_3 = \frac{1}{16}\frac{T\ell^2}{EI}$
(5)		$\theta_1 = \theta_2 = \frac{1}{16}\frac{F\ell^2}{EI}$; $w_3 = \frac{1}{48}\frac{F\ell^3}{EI}$
(6)		$\theta_1 = \theta_2 = \frac{1}{24}\frac{q\ell^3}{EI}$; $w_3 = \frac{5}{384}\frac{q\ell^4}{EI}$
(a)		$\theta_1 = \theta_2 = \frac{1}{24}\frac{T\ell}{EI}$; $\theta_3 = \frac{1}{12}\frac{T\ell}{EI}$; $w_3 = 0$

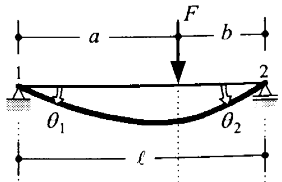
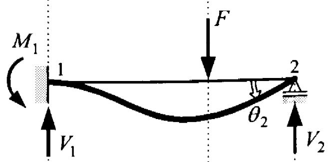
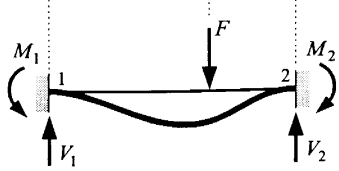
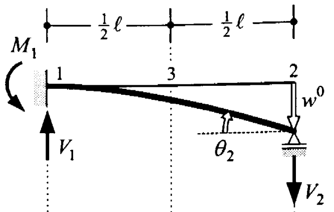
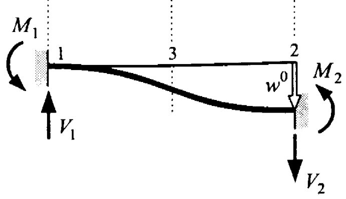
statisch onbepaalde ligger (enkelzijdig ingeklemd)

(7)		$\theta_2 = \frac{1}{4}\frac{T\ell}{EI}$; $w_3 = \frac{1}{32}\frac{T\ell^2}{EI}$ $M_1 = \frac{1}{2}T$; $V_1 = V_2 = \frac{3}{2}\frac{T}{\ell}$
(8)		$\theta_2 = \frac{1}{32}\frac{F\ell^2}{EI}$; $w_3 = \frac{7}{768}\frac{F\ell^3}{EI}$ $M_1 = \frac{3}{16}F\ell$; $V_1 = \frac{11}{16}F$; $V_2 = \frac{5}{16}F$
(9)		$\theta_2 = \frac{1}{48}\frac{q\ell^3}{EI}$; $w_3 = \frac{1}{192}\frac{q\ell^4}{EI}$ $M_1 = \frac{1}{8}q\ell^2$; $V_1 = \frac{5}{8}q\ell$; $V_2 = \frac{3}{8}q\ell$

statisch onbepaalde ligger (tweezijdig ingeklemd)

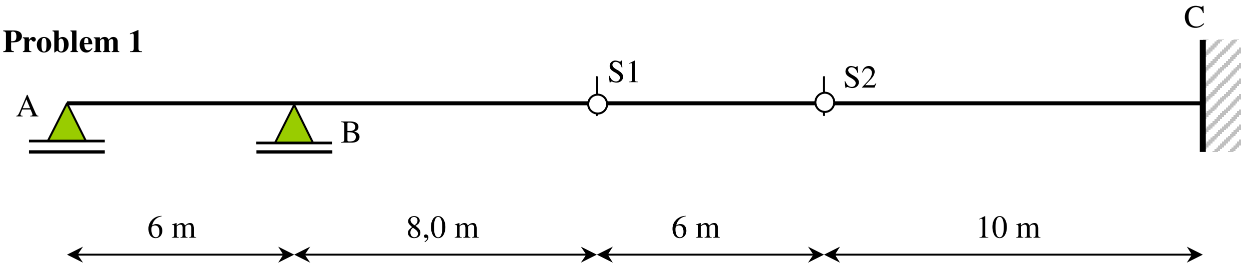
(10)		$w_3 = \frac{1}{192}\frac{F\ell^3}{EI}$ $M_1 = M_2 = \frac{1}{8}F\ell$; $V_1 = V_2 = \frac{1}{2}F$
(11)		$w_3 = \frac{1}{384}\frac{q\ell^4}{EI}$ $M_1 = M_2 = \frac{1}{12}q\ell^2$; $V_1 = V_2 = \frac{1}{2}q\ell$
(b)		$\theta_3 = \frac{1}{16}\frac{T\ell}{EI}$; $w_3 = 0$ $M_1 = M_2 = \frac{1}{4}T$; $V_1 = V_2 = \frac{3}{2}\frac{T}{\ell}$

Enkele formules voor prismatische liggers met buigstijfheid EI.
T, F en q zijn belastingen door resp. een koppel, kracht en gelijkmatig verdeelde belasting.
M_i en V_i zijn het buigend moment en de dwarskracht op einddoorsnede i van de ligger ten gevolge van de oplegreacties.

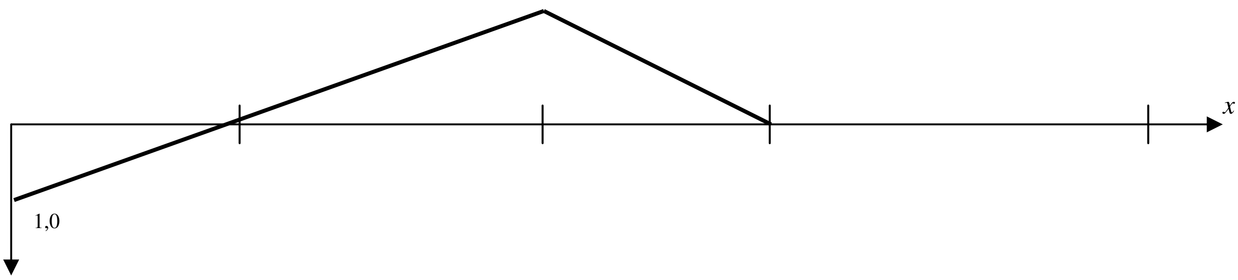
(c)		$\theta_1 = \dfrac{Fab(\ell+b)}{6EI\ell} = \dfrac{F\ell^2}{6EI}\left(2\dfrac{a}{\ell} - 3\dfrac{a^2}{\ell^2} + \dfrac{a^3}{\ell^3}\right)$ $\theta_2 = \dfrac{Fab(\ell+a)}{6EI\ell} = \dfrac{F\ell^2}{6EI}\left(\dfrac{a}{\ell} - \dfrac{a^3}{\ell^3}\right)$	
(d)		$M_1 = \dfrac{Fb(\ell^2-b^2)}{2\ell^2} = F\ell\left(\dfrac{a}{\ell} - \dfrac{3}{2}\dfrac{a^2}{\ell^2} + \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$ $V_1 = \dfrac{Fb(3\ell^2-b^2)}{2\ell^3} = F\left(1 - \dfrac{3}{2}\dfrac{a^2}{\ell^2} + \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$ $V_2 = \dfrac{Fa^2(3\ell-a)}{2\ell^3} = F\left(\dfrac{3}{2}\dfrac{a^2}{\ell^2} - \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$ $\theta_2 = \dfrac{Fa^2b}{4EI\ell} = \dfrac{F\ell^2}{4EI}\left(\dfrac{a^2}{\ell^2} - \dfrac{a^3}{\ell^3}\right)$	*drie bij-de handjes*
(e)		$M_1 = \dfrac{Fab^2}{\ell^2} = F\ell\left(\dfrac{a}{\ell} - 2\dfrac{a^2}{\ell^2} + \dfrac{a^3}{\ell^3}\right)$ $V_1 = \dfrac{Fb^2(\ell+2a)}{\ell^3} = F\left(1 - 3\dfrac{a^2}{\ell^2} + 2\dfrac{a^3}{\ell^3}\right)$ $M_2 = \dfrac{Fa^2b}{\ell^2} = F\ell\left(\dfrac{a^2}{\ell^2} - \dfrac{a^3}{\ell^3}\right)$ $V_2 = \dfrac{Fa^2(\ell+2b)}{\ell^3} = F\ell\left(3\dfrac{a^2}{\ell^2} - 2\dfrac{a^3}{\ell^3}\right)$	
(f)		$M_1 = \dfrac{3EI}{\ell^2}w^0;\quad V_1 = V_2 = \dfrac{3EI}{\ell^3}w^0$ $\theta_2 = \dfrac{3}{2}\dfrac{w^0}{\ell}$ $\theta_3 = \dfrac{9}{8}\dfrac{w^0}{\ell};\quad w_3 = \dfrac{5}{16}w^0$	*zettingen*
(g)		$M_1 = M_2 = \dfrac{6EI}{\ell^2}w^0;\quad V_1 = V_2 = \dfrac{12EI}{\ell^3}w^0$ $\theta_3 = \dfrac{3}{2}\dfrac{w^0}{\ell};\quad w_3 = \dfrac{1}{2}w^0$	

ANSWERS

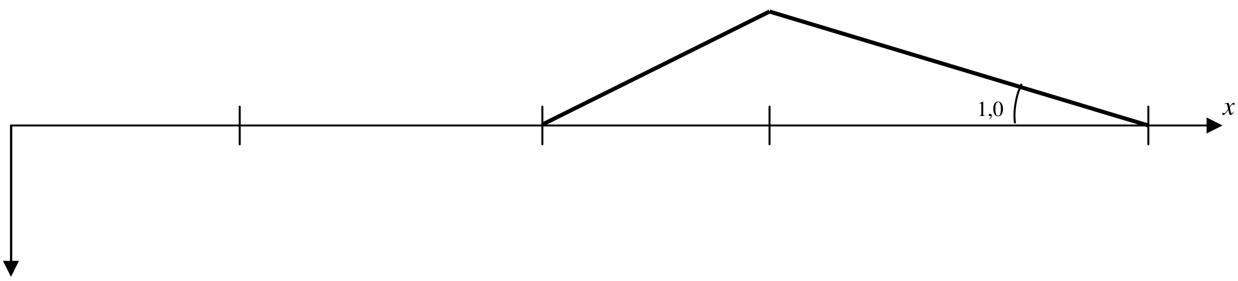
Problem 1



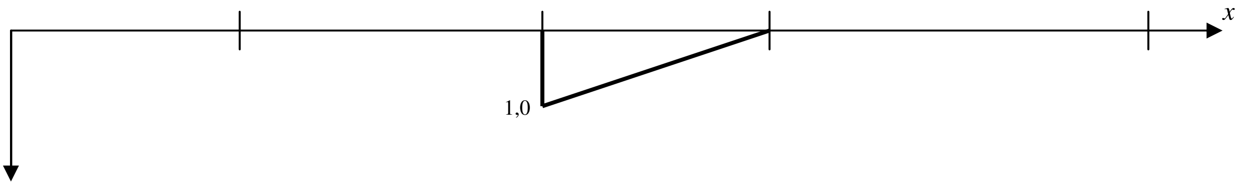
i-line
A_V



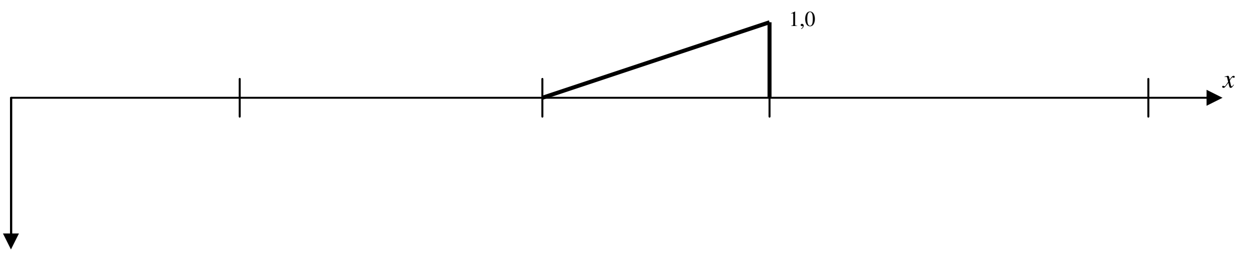
i-line
M_C



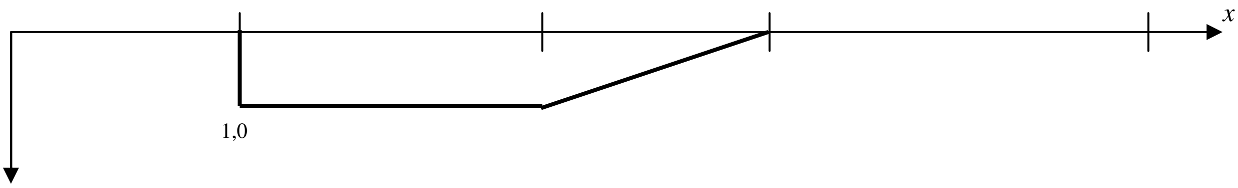
i-line
$V_{S1\text{-}L}$



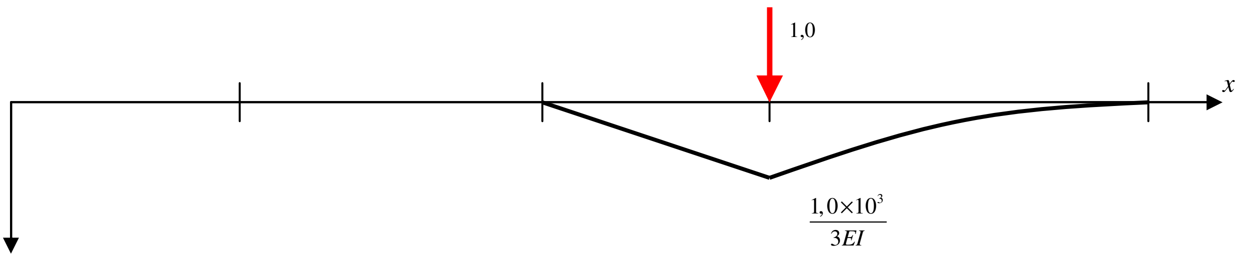
i-line
$V_{S2\text{-}R}$



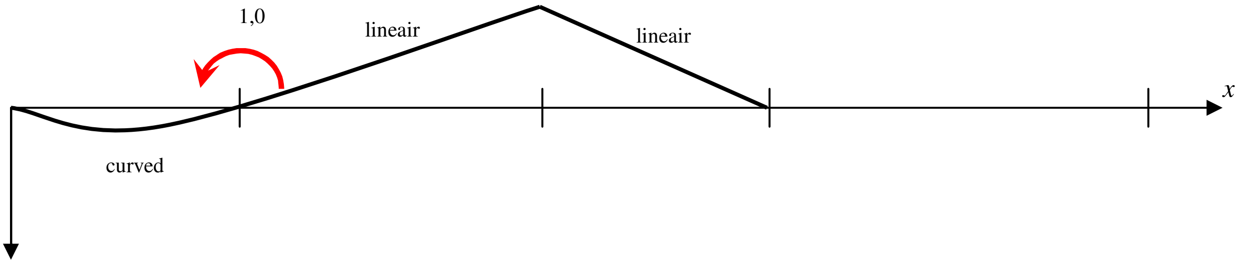
i-line
$V_{B\text{-}R}$



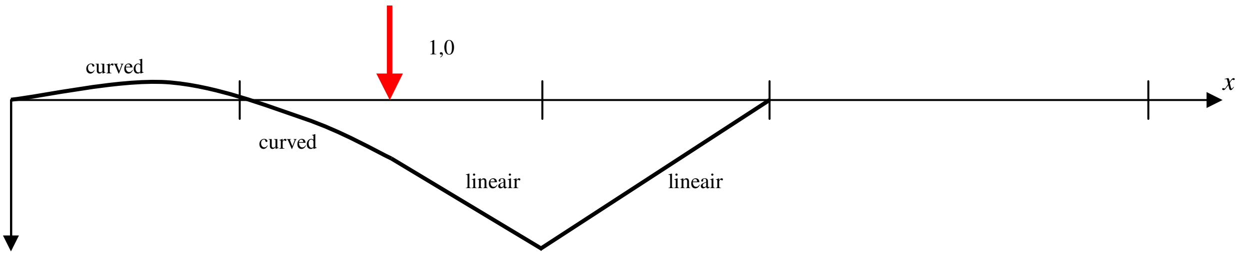
i-lijn
w_{S2}



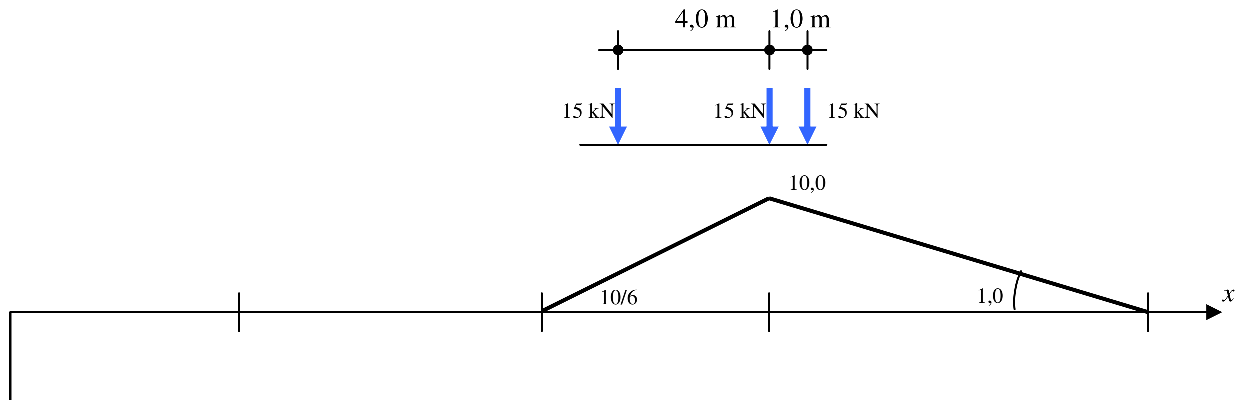
i-line
φ_B



i-line
$w_{S1\text{-}B}$

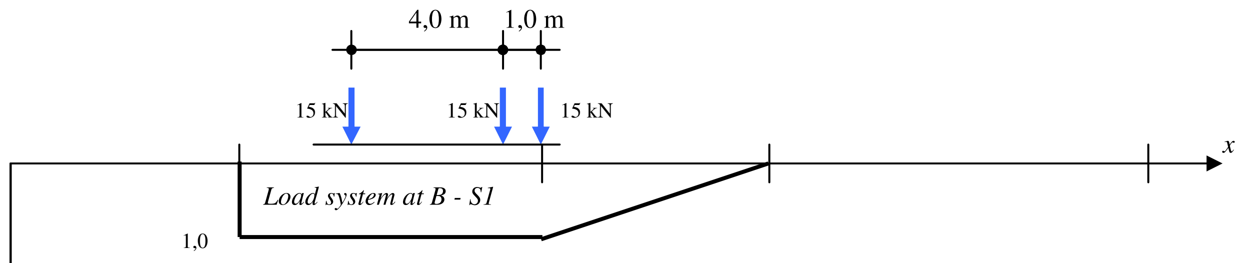


i-line
M_C



M_C = 335 kNm

i-line
$V_{B\text{-}R}$



V_B = 45 kN

Problem 2

a) Linear moment distribution at AB with a maximum of $5Fa$ at B. Subsequently returning to zero at C. The deformation symbol fort his entire part is $\bigcap$.

b) Assume a local coordinate system with the *x*-axis along the beam axis. Both member parts have an identical moment distribution:

$$M^{\text{AB}}(x) = -\frac{5Fa\,x}{5a} = Fx$$

The strain energy thus becomes:

$$E_v = 2\int_{x=0}^{x=5a} \frac{M^{\text{AB}}(x)^2}{2EI}\,dx = \frac{125F^2a^3}{3EI}$$

c) Differentiate this energy with respect to the load in order to obatin the displacement according to Castigliano's principle:

$$w = \frac{\mathrm{d}E_v}{\mathrm{d}F} = \frac{250Fa^3}{3EI}$$

d) To find the rotation use a dummy couple at B. The moment distribution has tob e adjusted fort his additional couple at B. Assume an anti clock wise couple in order to find a positive rotation according to the definitions of positive rotations in the *x*-*z* plane:

$$M^{\text{AB}}(x) = -\frac{5Fa\,x}{5a} + \frac{Tx}{5a} = Fx + \frac{Tx}{5a}$$

$$M^{\text{CB}}(x) = -\frac{5Fa\,x}{5a} = Fx$$

The rotation at B thus becomes:

$$\varphi_{\text{B}} = \frac{\mathrm{d}E_V}{\mathrm{d}T} = \frac{\mathrm{d}}{\mathrm{d}T}\left[\int_{x=0}^{x=5a} \frac{M^{\text{AB}}(x)^2}{2EI}\,\mathrm{d}x + \int_{x=0}^{x=5a} \frac{M^{\text{CB}}(x)^2}{2EI}\,\mathrm{d}x\right] =$$

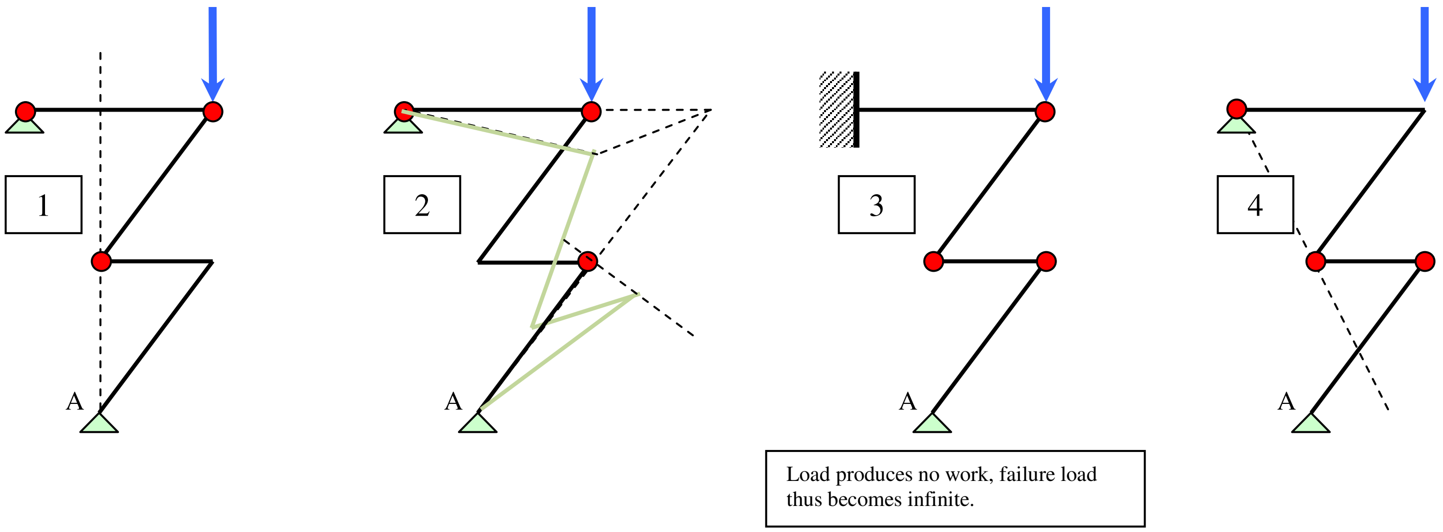
$$= \frac{5Ta}{3EI} - \frac{25Fa^2}{3EI}$$

We are only interested in the rotation duet o the actual load. For a zero dummy couple *T* we obtain the required answer:

$$\varphi_{\text{B}} = -\frac{25Fa^2}{3EI}$$

Problem 3

The structure is two-fold statically indeterminate. We therefore need three hinges for a (plastic) mechanism. There are four possible positions for hinges. This results in four possible mechanisms to investigate. In only three cases the load can produce work which results in three mechanisms which have to be investigated. Small sketches of these mechanisms are shown below.



Elaborate each possible mechanism:

1. Show clearly the kinematics of each bar with respect to all the others,
2. Add the correct direction of all moments at the plastic hinges,
3. Present a correct virtual work equation.

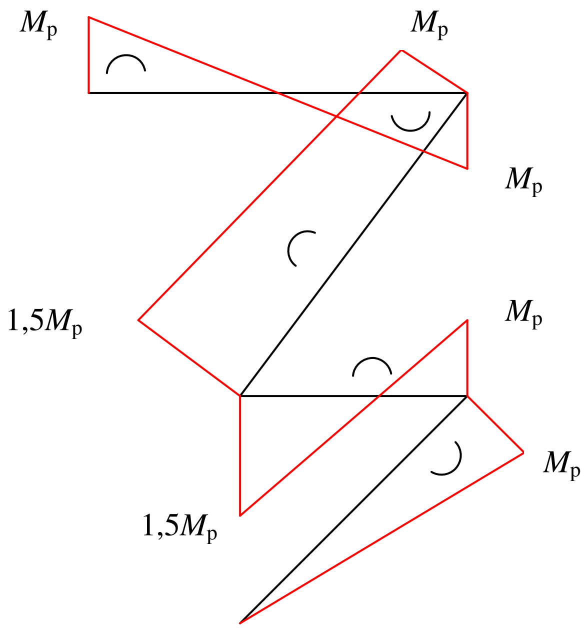
If the kinematics of a mechanism is incorrect no credits will be given to the solution of this mechanism since it is a vital part of the solution strategy and therefore an essential learning outcome to assess.

1: $$F_p = \frac{17M_p}{9a} = 1{,}889\frac{M_p}{a}$$

2: $$F_p = \frac{37M_p}{30a} = 1{,}233\frac{M_p}{a}$$

4: $$F_p = \frac{23M_p}{15a} = 1{,}533\frac{M_p}{a}$$

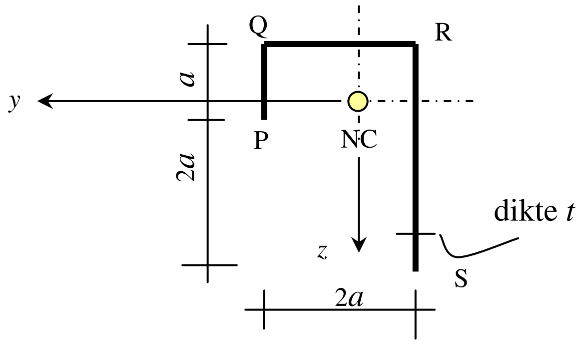
Mechanism 2 leads to the lowest failure load and is the failure mechanism.
The moment distribution can be drawn almost completely without calculations. Only the moment at D is yet unknown. First find the support reactions and then compute the moment at D. The resulting moment is smaller than the strength $2M_p$ at D so the results satisfy Pragers uniqueness theoreme.



If the moment distribution at no point exceeds the strength of the structure for the mechanism with the smallest failure or collapse load then this mechanism is the collapse mechanism and the load found is the collapse load.

Problem 4

a) The position of the NC can be found e.g. with respect to the upper right corner R with:



$$y_{NC} = \frac{(t \times a \times 2 \times a + t \times 2 \times a \times a)}{(6 \times t \times a)} = \frac{2}{3}a = 120 \text{ mm}$$

$$z_{NC} = \frac{(t \times a \times \frac{1}{2}a + t \times 3 \times a \times \frac{3}{2}a)}{(6 \times t \times a)} = \frac{5}{6}a = 150 \text{ mm}$$

The cross sectional properties become:

$$EI_{yy} = E \times (\text{t}a \times (\tfrac{4}{3}a)^2 + \tfrac{1}{12}t \times (2a)^3 + 2ta \times (\tfrac{1}{3}a)^2 + 3ta \times (\tfrac{2}{3}a)^2) = 4ta^3 \times E$$

$$EI_{zz} = E \times (\tfrac{1}{12}ta^3 + ta \times (\tfrac{1}{3}a)^2 + 2ta \times (\tfrac{5}{6}a)^2 + \tfrac{1}{12}t \times (3a)^3 + 3ta \times (\tfrac{2}{3}a)^2) = \frac{31ta^3}{6} \times E$$

$$EI_{yz} = E \times (ta \times (\tfrac{4}{3}a) \times (-\tfrac{1}{3}a) + 2at \times (\tfrac{1}{3}a) \times (-\tfrac{5}{6}a) + 3at \times (-\tfrac{2}{3}a) \times (\tfrac{2}{3}a)) = -\frac{7ta^3}{3} \times E$$

b) The moment at the cross section is:

$$M = \begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4}Fl \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \times 10^8 \end{bmatrix} \quad \text{Nmm}$$

The curvature can be found with:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}$$

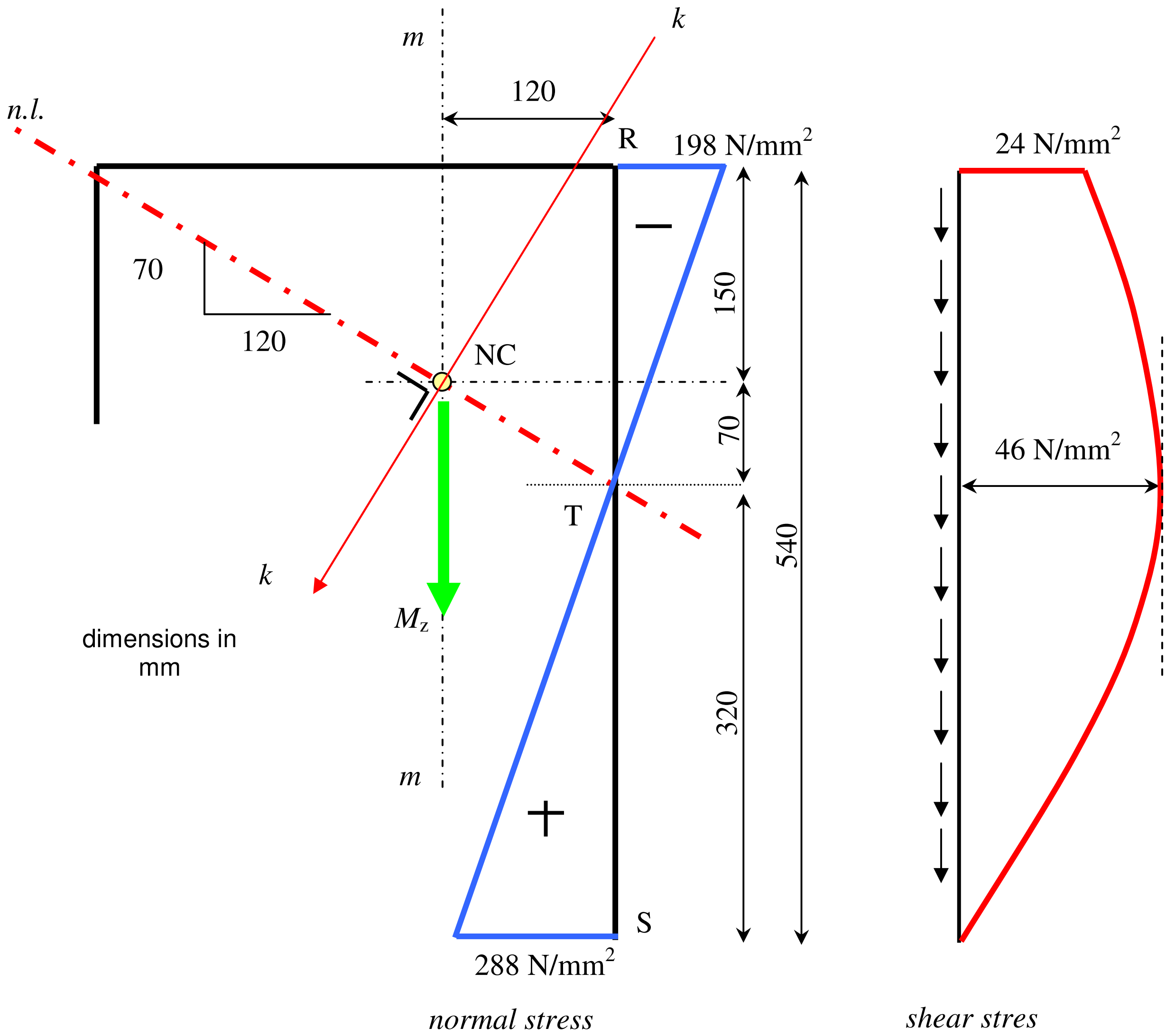
Which results in:

$$\begin{bmatrix} 0 \\ 2 \times 10^8 \end{bmatrix} = 2 \times 10^5 \begin{bmatrix} 23328 \times 10^4 & -13608 \times 10^4 \\ -13608 \times 10^4 & 30132 \times 10^4 \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}$$

Elaborating yields to:

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \begin{bmatrix} 0{,}2628 \\ 0{,}4506 \end{bmatrix} \times 10^{-5}$$

The plane of loading *m-m* and plane of curvature k-k are shown in the graph on the next page.



c) The equation of the neutral axis becomes:

$$\kappa_y \times y + \kappa_z \times z = 0$$

Since no normal force acts at the cross section, the n.a. runs through the NC and intersects the coordinate system at y = -120 mm and z = 70 mm. This is also shown in the figure above. The n.a. can be simplified to:

$$70y + 120z = 0$$

d) The normal stress can be found with:

$$\sigma(y,z) = E \times \varepsilon(y,z) = E \times (\kappa_y \times y + \kappa_z \times z)$$

The stress ditribution is in plane. For the part RS the stress distribution can be visualised with a straight line. To obtain this line stresses have tob e computed at R and S only.

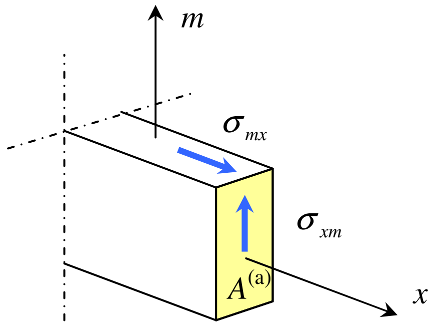
point	y	z	Stress [N/mm^2]
R	-120	-150	-198,25
S	-120	390	288,37

e) The shear stress distribution is parabolic if the normal stress is lineair. The maximum shear stress will occur at the points of intersection of the neutral axis and the cross section. To find the distribution of the shear stresses for RS the following three points have to be taken into consideration:

- S, shear stress is zero by definition
- T, maximum shear stress, compute this
- R tob e computed

The cross section is unsymmetrical. The computations have to be executed with the general solution strategy:

$$s_x^{(a)} = -\frac{R_M^{(a)}}{M} V; \quad \sigma_{xm} = \frac{s_x^{(a)}}{b^{(a)}}$$



In this R_M is the resultant force at the sliding element $A^{(a)}$ resulting from the the normal stress distribution duet o bending only. The moment and shear in this expression are the resulting moment and shear in the cross section:

$$M = \sqrt{M_y^2 + M_z^2} = 200 \times 10^6 \text{ Nmm}$$
$$V = \sqrt{V_y^2 + V_z^2} = 200 \times 10^3 \text{ N}$$

Both moment and shear are positive and act in the same plane of loading *m-m*. The resulting shear stress will have a direction which is downwards which makes sense due to the positive direction of the shear force on a positive section.

Starting with point T we find:

$$\sigma_{xm} = -\frac{\frac{1}{2} \times 288,366 \times 320 \times 10}{10 \times 200 \times 10^6} \times 200 \times 10^3 = -46 \text{ N/mm}^2$$

For the sliding element used a positive answer results in a shear stress which is upwards. So the result is in line with the expected direction of the shear stress.

For point R we find:

$$\sigma_{xm} = -\frac{10 \times (\frac{1}{2} \times 288,366 \times 320 - \frac{1}{2} \times 198,252 \times 220)}{10 \times 200 \times 10^6} \times 200 \times 10^3 = -24 \text{ N/mm}^2$$

The results can also be found in the graph on the previous page.