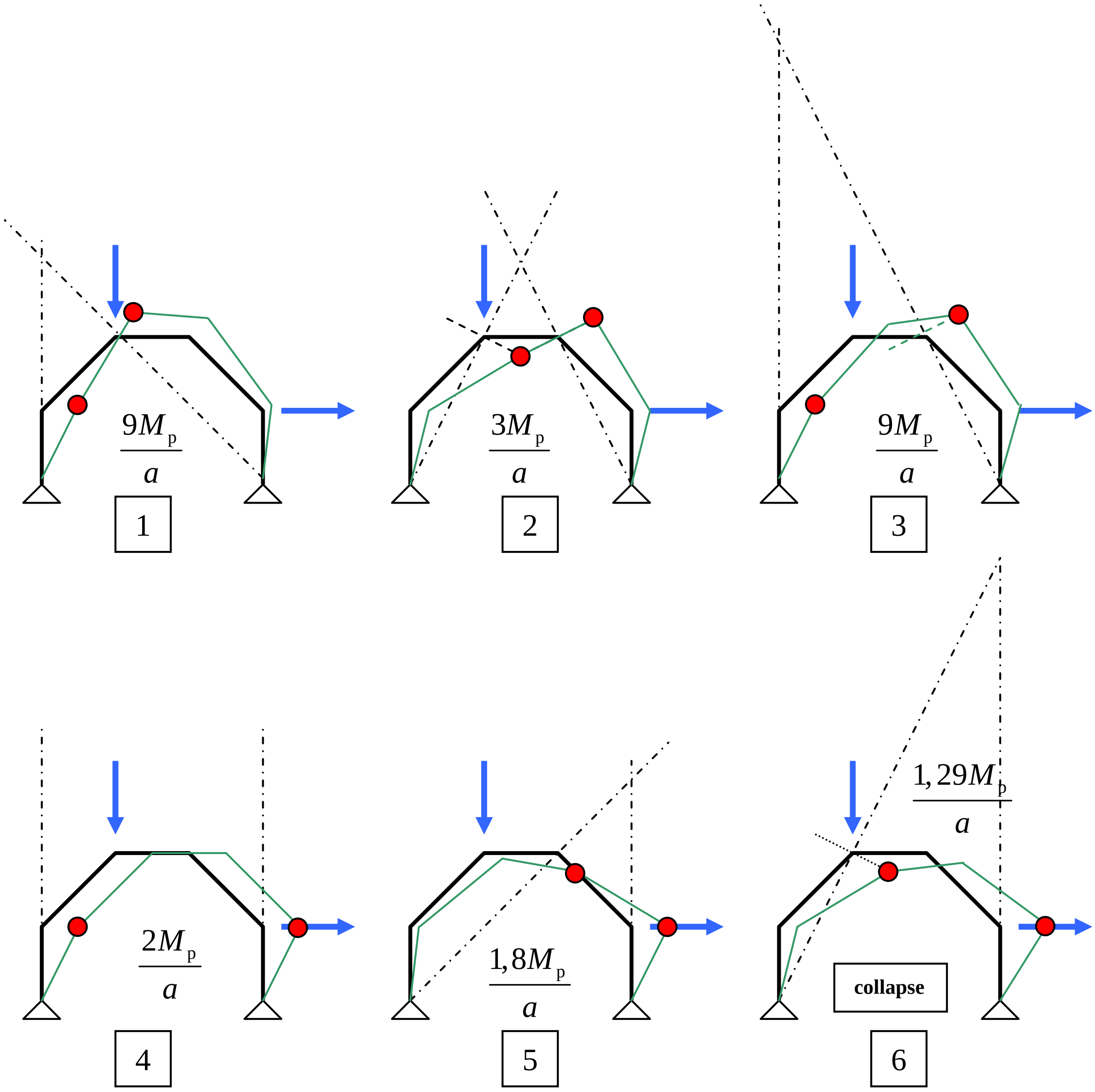
ANSWERS

Problem 1 : Plasticity

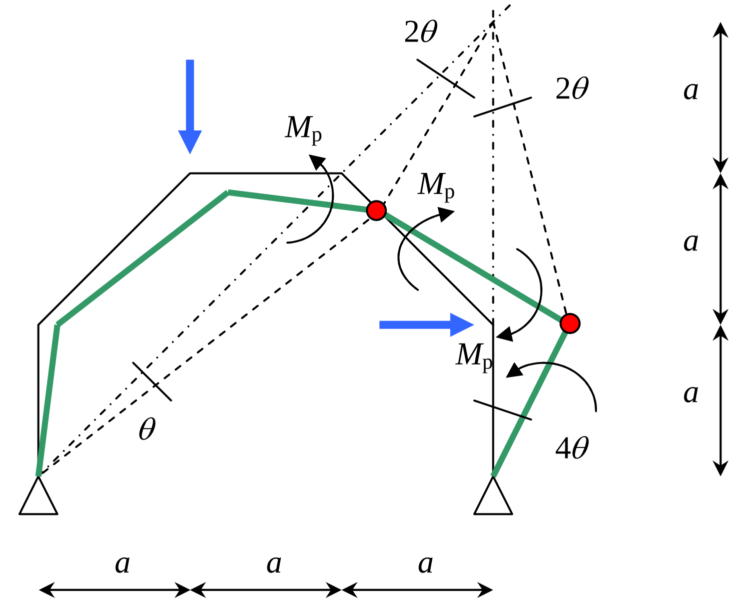
a) The structure is statically indeterminate to the degree of one. We therefore need two hinges for a (plastic) mechanism. There are four possible positions for hinges. This results in 6 possible mechanisms to investigate. Small sketches are given below.



From these sketches it is possible to deduce the most relevant mechanisms. The sway mechanism nr 4 is always very sensitive. Also mechanisms for which both the displacement and load are pointing in the same direction are most likely to occur. This results in mechanism 2,4,5 and 6. This will lead to the smallest collapse load. So we will only investigate these latter three mechanisms.

b) Mechanism 4 is very simple. From the sketch follows directly $F_p = \dfrac{2M_p}{a}$

Mechanism nr 5 is shown in detail below.

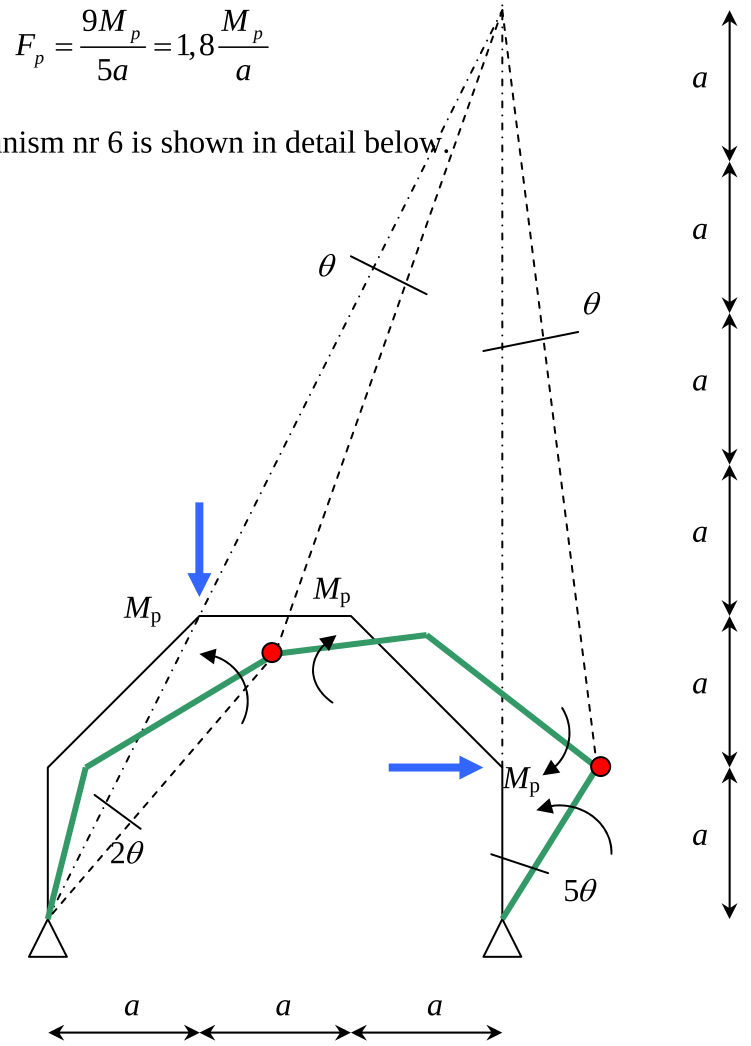


From this we can find the corresponding load using virtual work:

$$\delta A = -M_p\delta\theta - M_p 2\delta\theta - M_p 2\delta\theta - M_p 4\delta\theta + Fa\delta\theta + Fa4\delta\theta = 0$$

$$F_p = \frac{9M_p}{5a} = 1{,}8\frac{M_p}{a}$$

Mechanism nr 6 is shown in detail below.



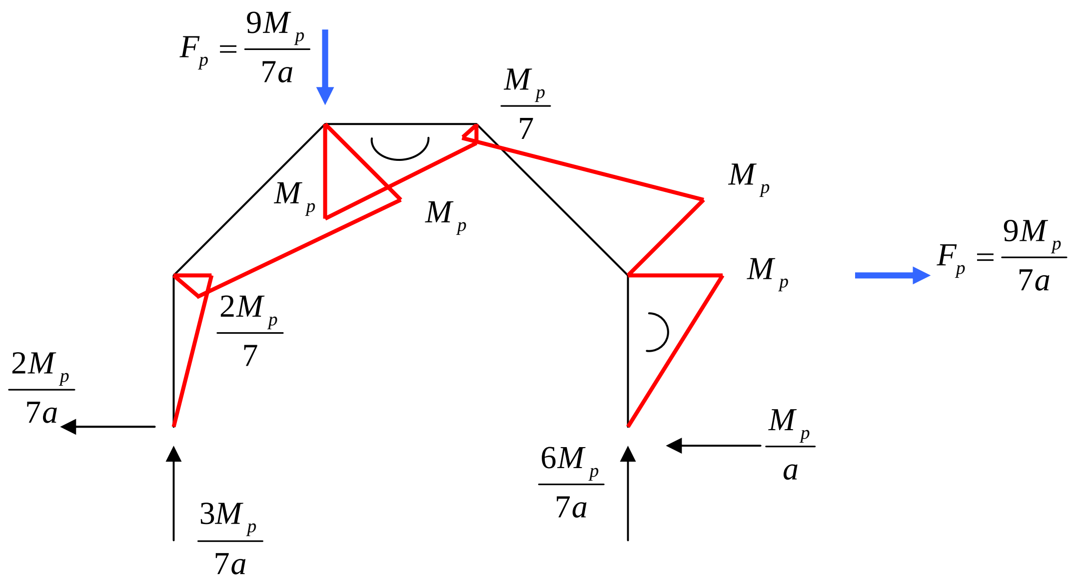
From this we can find the corresponding load using virtual work:

$$\delta A = -M_p 2\delta\theta - M_p \delta\theta - M_p \delta\theta - M_p 5\delta\theta + Fa2\delta\theta + Fa5\delta\theta = 0$$

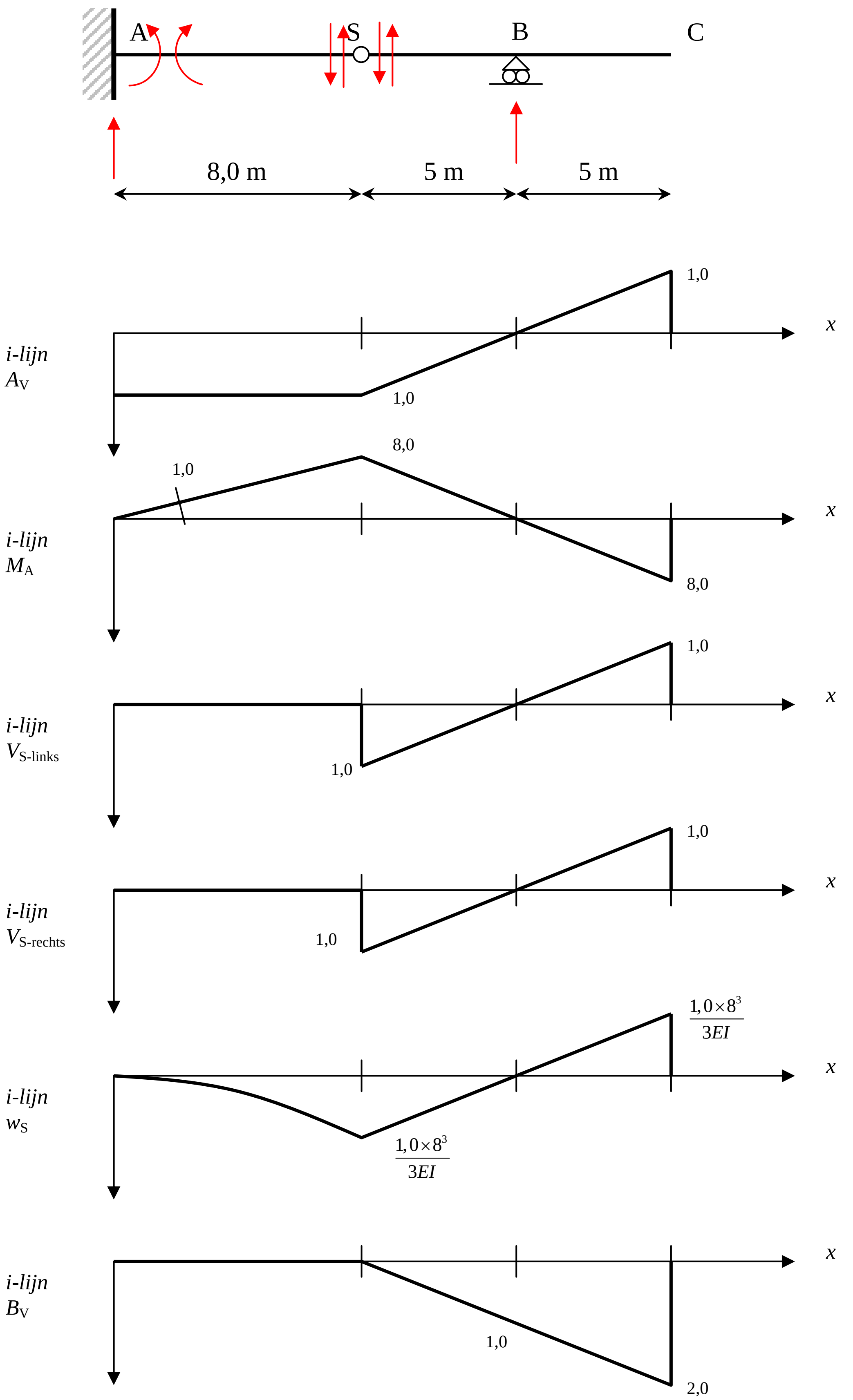
$$F_p = \frac{9M_p}{7a} = 1,28\frac{M_p}{a}$$

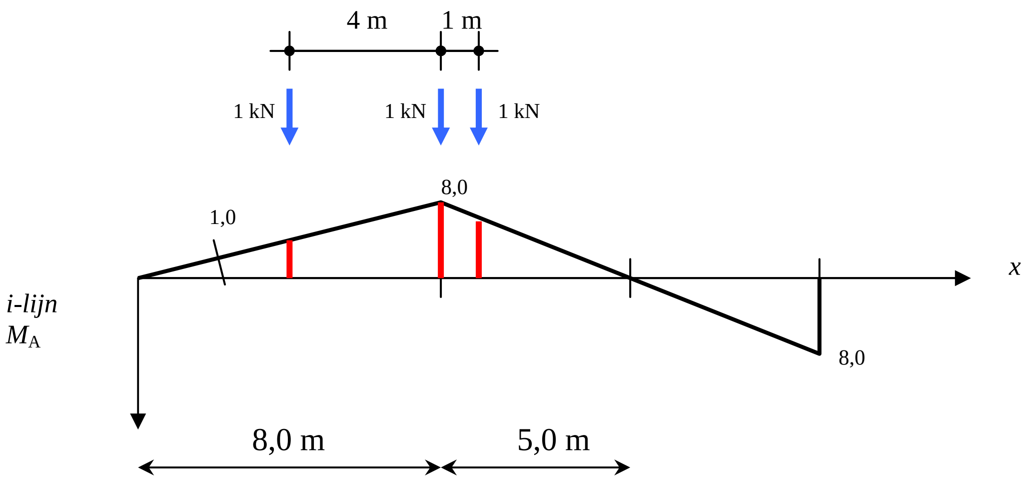
Mechanism 6 results in the smallest load for which a mechanism occurs. This load is therefore the collapse load.

c) The moment distribution at collapse can be found based on equilibrium.

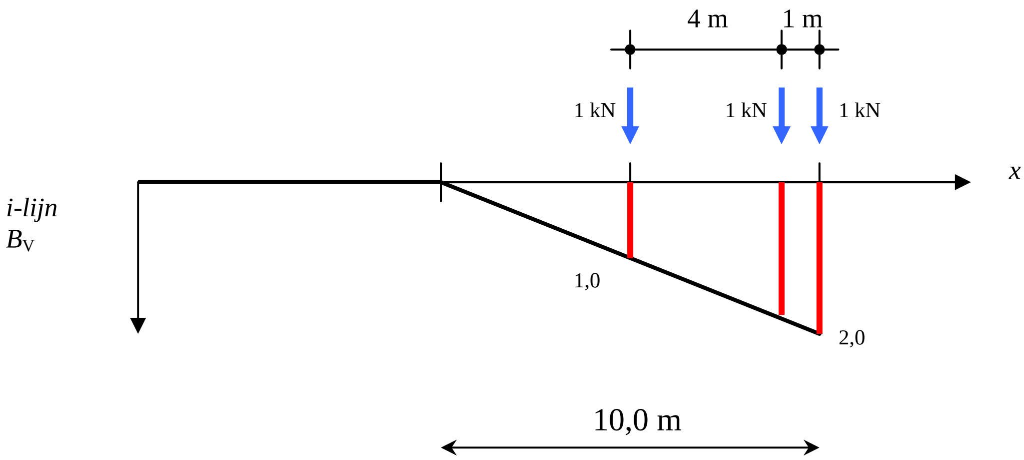


Problem 2 : Influence lines





$$M_A = -4,0 \times F - 8,0 \times F - \tfrac{4}{5} \times 8,0 \times F = -18,4F = -18,4 \quad \text{kNm}$$



$$B_V = 1,0 \times F + 1,8 \times F + 2,0 \times F = 4,8F = 4,8 \quad \text{kN}$$

Problem 3 : Work and Energy Methods

The structure is statically indeterminate (n = 1). If we consider the normal force in the two-force-member BC as the fundamental unknown, both the bending moment distribution for element ACD and the normal force distribution for BC can be computed:

$$M^{AC}(x) = \begin{cases} -F(3l-x) + N(2l-x) & 0 < x < 2l \\ -F(3l-x) & 2l < x < 3l \end{cases}$$

$$N^{BS}(x) = N$$

Minimum work (deformation energy) requires: (one of the options ….)

$$\frac{\partial E_v}{\partial N} = 0 \quad \text{with:} \quad E_v = \int_0^{3l} \frac{M^2}{2EI} dx + \int_0^{l} \frac{N^2}{2EA} dx$$

To simplify the hand calculation we use:

$$\frac{\partial E_v}{\partial N} = \frac{\partial \int_0^{3l} \frac{M^2}{2EI} dx + \int_0^{l} \frac{N^2}{2EA} dx}{\partial N} = \int_0^{3l} \frac{M}{EI}\frac{\partial M}{\partial N} dx + \int_0^{l} \frac{N}{EA}\frac{\partial N}{\partial N} dx = \int_0^{3l} \frac{M}{EI}\frac{\partial M}{\partial N} dx + \frac{Nl}{EA}$$

This results in:

$$\frac{\partial E_v}{\partial N} = \int_0^{3l} \frac{M}{EI}\frac{\partial M}{\partial N} dx + \frac{Nl}{EA} = 0; \quad \text{with:} \quad \frac{\partial M}{\partial N} = \begin{cases} 2l - x & 0 < x < 2l \\ 0 & 2l < x < 3l \end{cases}$$

Elaborating this latter expression results in:

$$\frac{1}{EI}\int_0^{2l} \left[-F(3l-x) + N(2l-x)\right](2l-x)dx + \frac{1}{EI}\int_{2l}^{3l} \left[-F(3l-x)\right]0dx + \frac{Nl}{EA} = 0$$

$$\frac{-F}{EI}(6l^2 - 5lx + x^2)\Big|_0^{2l} + \frac{N}{EI}(4l^2 - 4lx + x^2)\Big|_0^{2l} + \frac{Nl}{EA} = 0$$

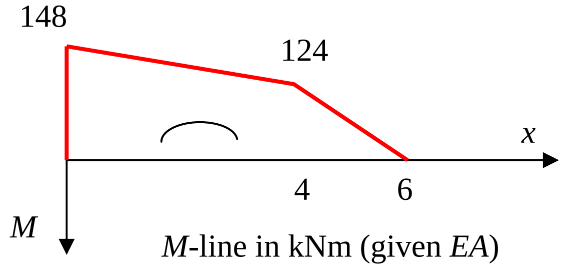
$$\frac{-F}{EI}(12l^3 - 10l^3 + \tfrac{8}{3}l^3) + \frac{N}{EI}(8l^3 - 8l^3 + \tfrac{8}{3}l^3) + \frac{Nl}{EA} = 0$$

$$N\left(\frac{8l^3}{3EI} + \frac{l}{EA}\right) = \frac{14Fl^3}{3EI} \quad \Leftrightarrow \quad N = \frac{\frac{14Fl^3}{3EI}}{\frac{8l^3}{3EI} + \frac{l}{EA}} = \frac{14Fl^2EA}{8l^2EA + 3EI} = 56 \text{ kN}$$

For infinite EA the resulting normal force in the two-force-member becomes:

$$N = \frac{14Fl^2EA}{8l^2EA + 3EI} = \frac{14Fl^2}{8l^2 + \frac{3EI}{EA}}$$

$$N_{\lim EA\to\infty} = \frac{14F}{8} = 1{,}75F = 108{,}5 \text{ kN}$$



M-line in kNm (given EA)

OPGAVE 4 : Niet-symmetrische doorsnede

The axial stiffness of the cross section can be found with:

$$EA = E\times(3a\times 2a)+2E\times(a\times 2a)+6E\times(a\times 4a)=34Ea^2$$

The origin of the coordinate system used is located at the NC. The vertical position of the NC with respect to the upper side of the cross section is:

$$\Delta z_{NC}=\frac{E\times(3a\times 2a\times a)+2E\times(a\times 2a\times 2\tfrac{1}{2}a)+6E\times(a\times 4a\times 2\tfrac{1}{2}a)}{EA}=\frac{38}{17}a=111,7 \text{ mm}$$

The horizontal position with respect to the left side of the cross section is:

$$\Delta y_{NC}=\frac{E\times(3a\times 2a\times \tfrac{3}{2}a)+2E\times(a\times 2a\times a)+6E\times(a\times 4a\times 4a)}{EA}=\frac{109}{34}a=160,3 \text{ mm}$$

Only one bending stiffness was asked for:

$$\begin{aligned}EI_{yz} =\ & E\times\left[3a\times 2a\times(y_{NC}-1,5a)\times(-z_{NC}+a)\right]+\\ & 2E\times\left[a\times 2a\times(y_{NC}-a)\times(-z_{NC}+2,5a)\right]+\\ & 6E\times\left[a\times 4a\times(y_{NC}-4a)\times(-z_{NC}+2,5a)\right]=-\frac{261Ea^4}{17}=-1566a^2\times\frac{Ea^2}{102}=-1439,3\times 10^9 \text{ Nmm}^2\end{aligned}$$

The *cross sectional constitutive relation* relates the sectional forces to the deformations of the cross section:

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix}=\begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{yz} & EI_{zz} \end{bmatrix}\begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}$$

cross sectional constitutive relation:

$$\frac{Ea^2}{102}\begin{bmatrix} 3468 & 0 & 0 \\ 0 & 9169a^2 & -1566a^2 \\ 0 & -1566a^2 & 1576a^2 \end{bmatrix}=\begin{bmatrix} 1275\times 10^6 & 0 & 0 \\ 0 & 8227\times 10^9 & -1439\times 10^9 \\ 0 & -1439\times 10^9 & 1449\times 10^9 \end{bmatrix}$$

Since this structure is loaded in bending only, the strain ε at the NC must be zero. The curvatures can be found with the constitutive relation:

$$\varepsilon = \frac{N}{EA}=0$$

$$\kappa_y=\frac{1}{EI_{yy}EI_{zz}-EI_{yz}^2}\left(EI_{zz}\times M_y - EI_{yz}\times M_z\right)=1,0\times 10^{-5}$$

$$\kappa_z=\frac{1}{EI_{yy}EI_{zz}-EI_{yz}^2}\left(-EI_{yz}\times M_y + EI_{yy}\times M_z\right)=1,0\times 10^{-5}$$

The direction of the *plane of loading* and the *plane of curvature* can be obtained with:

$$\tan\alpha_m=\frac{M_z}{M_y}=0 \quad\Rightarrow\quad \alpha_m=0^\circ; \qquad \tan\alpha_k=\frac{\kappa_z}{\kappa_y}=1,0 \quad\Rightarrow\quad \alpha_k=45^\circ$$

The stresses for each point of the cross section can be computed with:

$$\sigma(y,z) = E \times (\varepsilon + \kappa_y \times y + \kappa_z \times z) \quad \text{N/mm}^2$$

The neutral axis *n.a.* can also be found with this latter expression by:

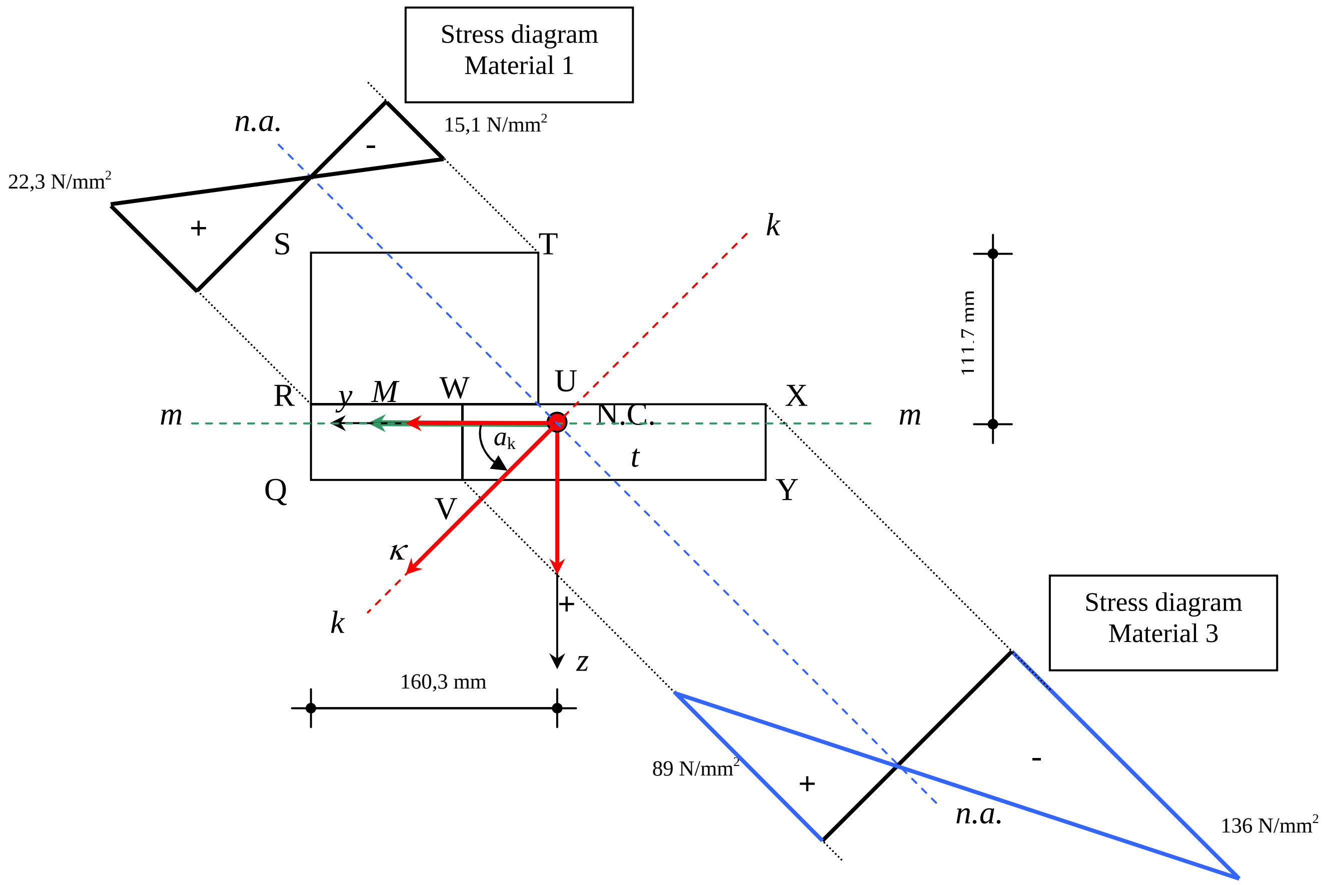
$$\varepsilon(y,z) = \varepsilon + \kappa_y \times y + \kappa_z \times z = 0 \quad \Leftrightarrow$$
$$\kappa_y \times y + \kappa_z \times z = 0$$

The stress distribution can be visualized with a few points. Only the most outer points with respect to the neutral axis are needed (material 1 : T and R; material 3 : X and V). Points S and U are added for the next question.

Tabel : Stress in the specified points

Material	point	*y* [mm]	*z* [mm]	*E* [N/mm^2]	*Stress* [N/mm^2]
1	S	160,3	-111,8	15^{e}3	7,4
	R	160,3	-11,8	15^{e}3	22,3
	U	10,3	-11,8	15^{e}3	-0,2
	T	10,3	-111,8	15^{e}3	-15,1
3	V	60,3	38,2	90^{e}3	88,5
	X	-139,7	-11,8	90^{e}3	-136,3

The graphical presentation of the stress distribution at C is given in the next graph. The neutral axis goes through the NC since the normal force *N* is zero.



The moment *M* and thus the load *F* acts in the *x*-*m* plane. The curvature κ acts in the *x*-*k* plane. Due to the unsymmetrical cross section these planes do not coincide.

The shear flow in the interface RU can be obtained with the resulting force at the sliding element (a) with:

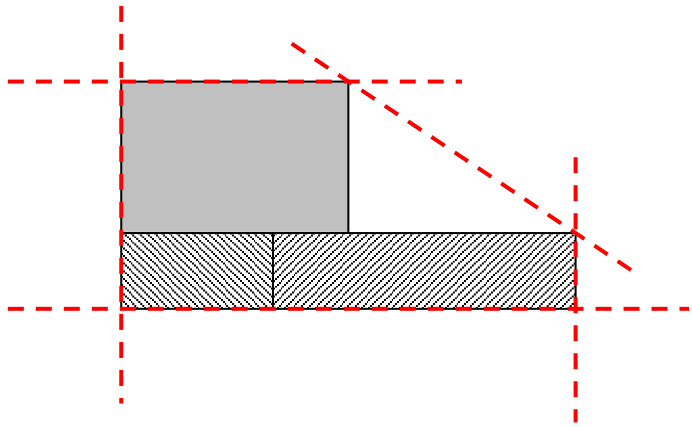
$$s_x = -\frac{R_M^{(a)}}{M} \times V = -\frac{\frac{1}{4}(\sigma_\mathrm{R} + \sigma_\mathrm{S} + \sigma_\mathrm{T} + \sigma_\mathrm{U}) \times 2a \times 3a}{M_y} \times V_y$$

$$s_x = -\frac{\frac{1}{4}(22,3 + 7,4 - 15,1 - 0,2) \times 100 \times 150}{70e6} \times 10e3 = -7,7 \text{ N/mm}$$

The kern of the cross section can be obtained with:

$$\begin{bmatrix} e_y \\ e_z \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{yz} & EI_{zz} \end{bmatrix} \begin{bmatrix} -1/y_1 \\ -1/z_1 \end{bmatrix}$$

For each *n.a.* which is just outside the cross section, one corner point of the kern can be found. For this cross section we have to investigate five positions of the *n.a.* as can be seen from the figure below.



The exact locations of the (corner) points of the kern were not asked for.