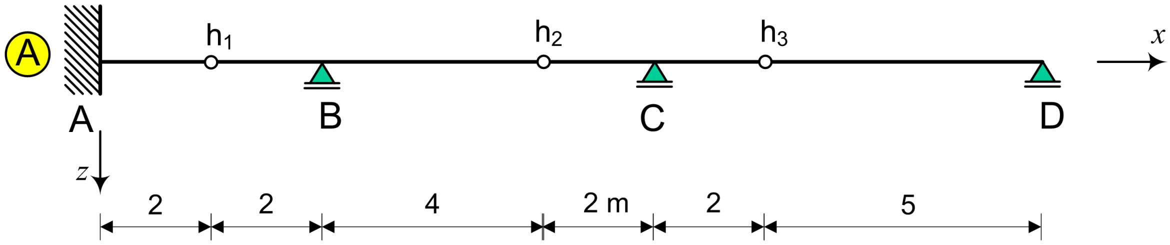
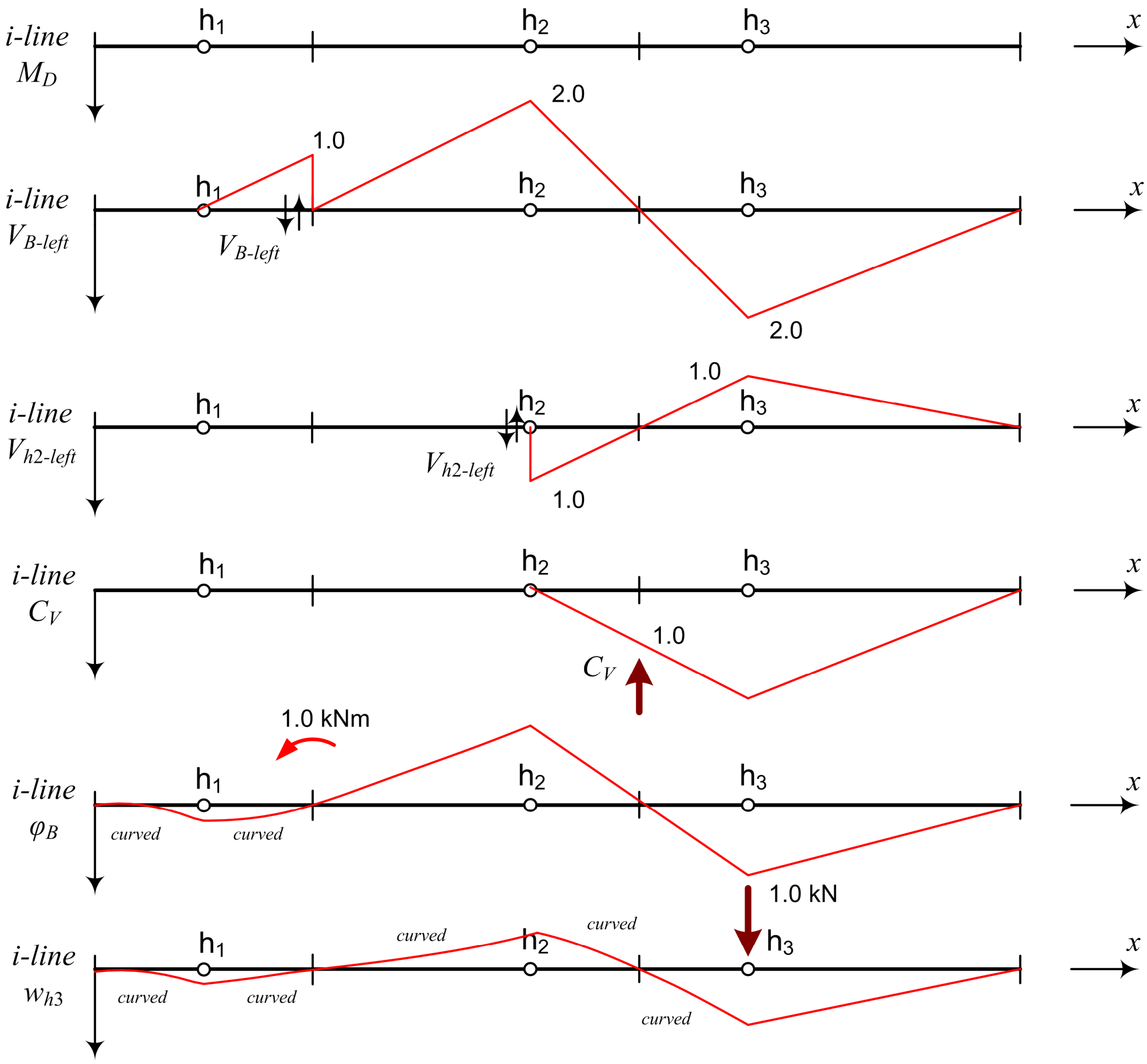
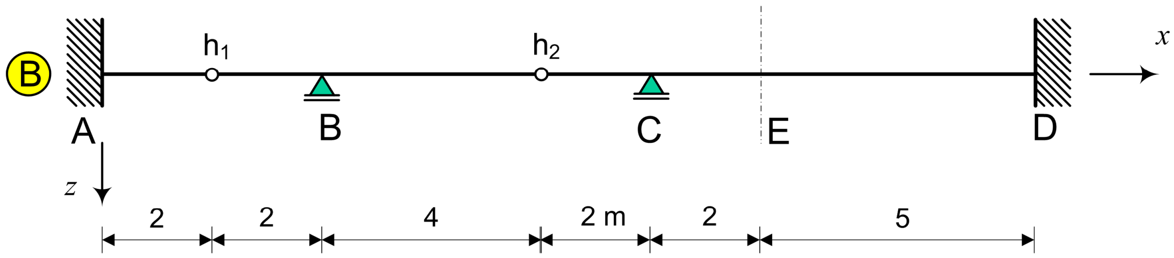
ANSWERS

Problem 1

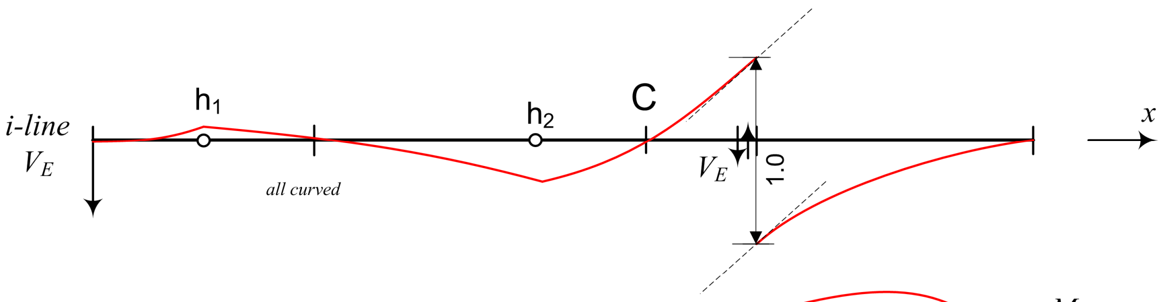


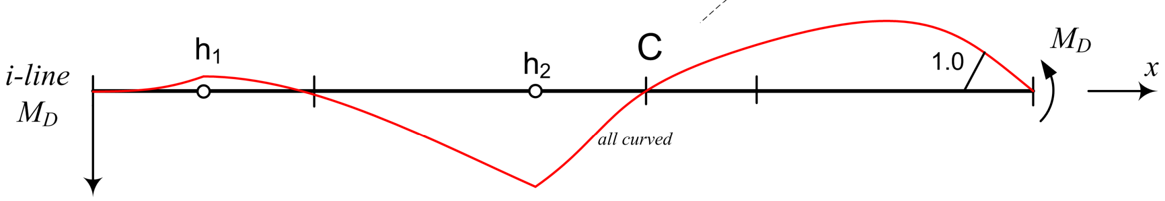
Answers beam (A):





Answers beam (B) :





Problem 2

a) In order to find the displacements, first of all the load distribution has to be obtained.
Since the structure is statically indeterminate (to the degree of 1) we could use the *force method* to solve the static unknown (also known as static redundant) with a deformation condition. This condition can be evaluated with an energy method. For instance take the moment at A as static redundant and use Castigliano's second theorem:

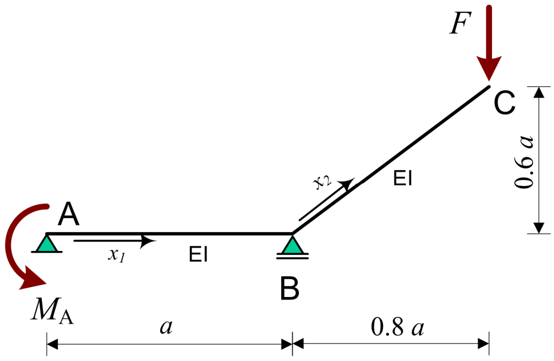
$$\frac{\mathrm{d}E_{\mathrm{c}}}{\mathrm{d}M_{\mathrm{A}}} = \varphi_{\mathrm{A}} = 0$$

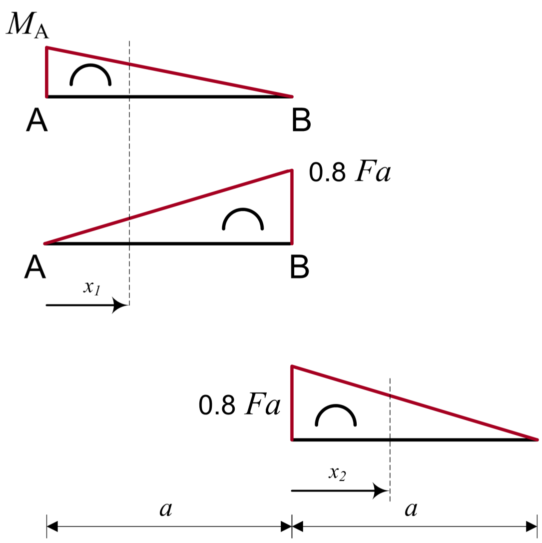
With the static redundant solved, the force distribution is known and based on this distribution with again the second theorem of Castigliano the vertical displacement at point C can be obtained:

$$w_{\mathrm{C}} = \frac{\mathrm{d}E_{\mathrm{c}}}{\mathrm{d}F}$$

b) The computation based upon the above sketched strategy results in the following steps:

Step 1: (obtain the force distribution)
Assume an unknown moment M_{A} at A (tension on the upper side of the beam).





The moment distribution for beam AB and BC with local axis along the beam axis results in:

beam AB $M^{\mathrm{AB}}(x_1) = -M_A\left(1-\frac{x_1}{a}\right) - \frac{4Fx_1}{5}$ $\frac{\mathrm{d}}{\mathrm{d}M_A}M^{\mathrm{AB}}(x_1) = -\left(1-\frac{x_1}{a}\right)$

beam BC $M^{\mathrm{BC}}(x_2) = -\frac{4Fa\left(1-\frac{x_2}{a}\right)}{5}$ $\frac{\mathrm{d}}{\mathrm{d}M_A}M^{\mathrm{BC}}(x_2) = 0$

Evaluating the deformation condition yields to:

$$\frac{\mathrm{d}E_c}{\mathrm{d}M_A} = \frac{\mathrm{d}}{\mathrm{d}M_A}\int_{ABC}\frac{M^2(x)}{2EI}\mathrm{d}x = \int_{ABC}\frac{M(x)\frac{\mathrm{d}M(x)}{\mathrm{d}M_A}}{EI}\mathrm{d}x =$$

$$= \int_{x_1=0}^{a}\frac{M^{\mathrm{AB}}(x_1)\frac{\mathrm{d}M^{\mathrm{AB}}(x_1)}{\mathrm{d}M_A}}{EI}\mathrm{d}x_1 + \int_{x_2=0}^{a}\frac{M^{\mathrm{BC}}(x_2)\frac{\mathrm{d}M^{\mathrm{BC}}(x_2)}{\mathrm{d}M_A}}{EI}\mathrm{d}x_2 =$$

$$= \int_{x_1=0}^{a}\frac{M^{\mathrm{AB}}(x_1)\frac{\mathrm{d}M^{\mathrm{AB}}(x_1)}{\mathrm{d}M_A}}{EI}\mathrm{d}x_1 = \int_{x_1=0}^{a}\frac{\left(-M_A\left(1-\frac{x_1}{a}\right)-\frac{4Fx_1}{5}\right)\left(-\left(1-\frac{x_1}{a}\right)\right)}{EI}\mathrm{d}x_1 =$$

$$= \frac{5aM_A + 2Fa^2}{15EI} = 0 \quad \Leftrightarrow M_A = -\tfrac{2}{5}Fa$$

Step 2 : (obtain displacement)

The obtained force distribution (moment distribution) can be written as:

beam AB $M^{\mathrm{AB}}(x_1) = \tfrac{2}{5}Fa\left(1-\frac{x_1}{a}\right) - \tfrac{4}{5}Fx_1$ $\frac{\mathrm{d}}{\mathrm{d}F}M^{\mathrm{AB}}(x_1) = \tfrac{2}{5}a\left(1-\frac{x_1}{a}\right) - \tfrac{4}{5}x_1$

beam BC $M^{\mathrm{BC}}(x_2) = -\tfrac{4}{5}Fa\left(1-\frac{x_2}{a}\right)$ $\frac{\mathrm{d}}{\mathrm{d}F}M^{\mathrm{BC}}(x_2) = -\tfrac{4}{5}a\left(1-\frac{x_2}{a}\right)$

The vertical displacement at C can be found with:

$$\frac{\mathrm{d}E_c}{\mathrm{d}F} = \frac{\mathrm{d}}{\mathrm{d}F}\int_{ABC}\frac{M^2(x)}{2EI}\mathrm{d}x = \int_{ABC}\frac{M(x)\frac{\mathrm{d}M(x)}{\mathrm{d}F}}{EI}\mathrm{d}x =$$

$$= \int_{x_1=0}^{a}\frac{M^{\mathrm{AB}}(x_1)\frac{\mathrm{d}M^{\mathrm{AB}}(x_1)}{\mathrm{d}F}}{EI}\mathrm{d}x_1 + \int_{x_2=0}^{a}\frac{M^{\mathrm{BC}}(x_2)\frac{\mathrm{d}M^{\mathrm{BC}}(x_2)}{\mathrm{d}F}}{EI}\mathrm{d}x_2 =$$

$$= \frac{28Fa^3}{75EI}$$

Problem 3

a) See reader
b) See reader
c) The structure is two-fold statically indeterminate. The maximum number of hinges to obtain a mechanism is three. These hinges may occur at four positions. Therefor a maximum of four mechanisms is to investigated:

   1 hinges at CDE, the load will not produce work, infinite failure load
   2 hinges at ACD
   3 hinges at ACE
   4 hinges at ADE

   So only the last three mechanisms have to be examined.

d) The failure load has to be found based on a correct mechanism with correct virtual rotations of all elements. The correct direction of the full plastic moments is also essential and finally a correct expression for the virtual work. This results in:

   ACD $2{,}33M_p/a$ *failure load*

   ACE $2{,}667M_p/a$

   ADE $4M_p/a$

e) The check is to construct the moment diagram for the failure load and check if Prager's uniqueness condition is met. At no position in the structure the moment can exceed the full plastic capacity.

Problem 4

a) This cross section is loaded in bending in the $x-z$ plane only. The plane of loading (*m-m*) follows from this. To find the plane of curvature first the position of the normal force centre NC and all cross sectional properties with respect to this NC have to be computed. With this results, the constitutive relation for the cross section can be obtained. With this relation and the cross sectional loading, the deformation at the cross section can be found. This results in the plane of curvature (*k-k*). From this also the position of the neutral axis *n.a.* can be found. According to the theory, the neutral axis must be perpendicular to the plane of curvature and will run through the NC.

For the cross section yields:

$$EA = 210\times10^3 \times\left(A_{tube} + a\times t + b\times t\right) = 210\times10^3 \times 9311.50 = 1955.42\times10^6 \text{ N}$$

Finding the NC: (with reference to the tube centre)

$$y_{NC} = \frac{210\times10^3 (a\times t\times(-R) + b\times t\times(-R-\tfrac{1}{2}b)}{EA} = -39.74 \text{ mm}$$

$$z_{NC} = \frac{210\times10^3 (a\times t\times\tfrac{1}{2}a + b\times t\times a)}{EA} = 34.67 \text{ mm}$$

The cross sectional properties (*double letter symbols*) for the composite cross section can be found as:

$$EI_{yy} = 210\times10^3 \times 10967.47\times10^4 \text{ Nmm}^2$$
$$EI_{zz} = 210\times10^3 \times 10767.15\times10^4 \text{ Nmm}^2$$
$$EI_{yz} = -210\times10^3 \times 3928.45\times10^4 \text{ Nmm}^2$$

The cross section at mid span is loaded in bending only:

$$N = 0 \text{ N}$$
$$M_y = 0 \text{ Nmm}$$
$$M_z = 36.75\times10^6 \text{ Nmm}$$

And the constitutive relation:

$$\begin{bmatrix} 0 \\ 0 \\ 36.75\times10^6 \end{bmatrix} = 210\times10^3 \begin{bmatrix} 93.1150\times10^2 & 0 & 0 \\ 0 & 10967.47\times10^4 & -3928.45\times10^4 \\ 0 & -3928.45\times10^4 & 10767.15\times10^4 \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}$$

The deformation can be found:

$$\varepsilon = 0; \quad \kappa_y = 0.66969\times10^{-6}; \quad \kappa_z = 0.18696560\times10^{-5};$$

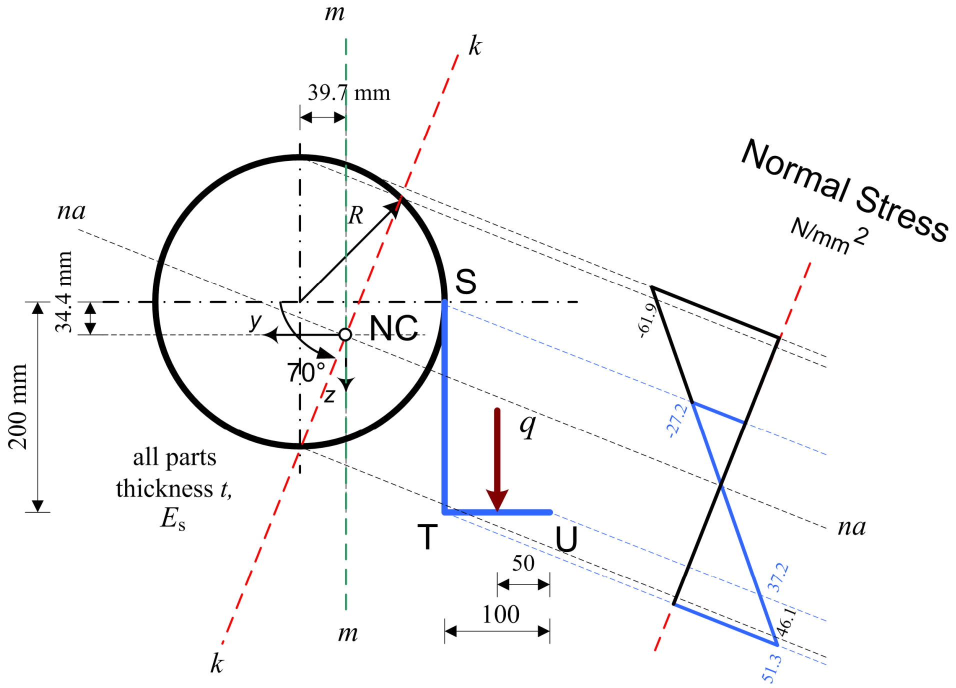
From this the directions of the *plane of loading* and the *plane of curvature* are:

$$\tan\alpha_k = \frac{\kappa_z}{\kappa_y} \quad \Rightarrow \quad \alpha_k = 70.3^\text{o}$$

$$\tan\alpha_m = \frac{M_z}{M_y} \Rightarrow \quad \alpha_m = 90.0^\text{o}$$

In this case the plane of loading *m-m* and the plane of curvature *k-k* differ by 20 degrees. The neutral axis *n.a.* can be expressed as:

$$0.6697\,y - 1.8697\,z = 0$$



Resulting *k-k*, *m-m* and *n.a*. due to given loads and normal stress distribution

b) The stresses can be found with the deformations:

$$\sigma(y,z) = E \times \left(\varepsilon + y \times \kappa_y + z \times \kappa_z\right)$$

With (y, z) as location with respect to the NC this results per point in:

point	y [mm]	z [mm]	E [N/mm^2]	σ [N/mm^2]
O top tube	39.74	-171.87	210000	-61.9
P lower tube	39.74	103.13	210000	46.1
S tube right	-97.76	-34.67	210000	-27.2
T lower corner	-97.76	165.63	210000	51.3
U free end	165.63	240	210000	37.2

These stresses are presented in the sketch above where the *k-k* line has been used to present the distribution of the normal stresses in the cross section for both materials.

c) The maximum shear stress at S in the L-section can be found with the resultant normal force $R_M^{(a)}$ based on the normal stress distribution due to bending only for the released part $A^{(a)}$ STU based on:

$$\tau_{\text{S-L}} = \frac{s_x^{(a)}}{t} = -\frac{R_M^{(a)}}{M}\frac{V}{t}$$

Where V is the maximum shear force of 42 kN at one of the supports and M is the resulting bending moment needed to find the resulting normal force R_M for a sliding part. For this we take either a unit moment (and subsequently the normal stress distribution due to this unit moment) or a cross section we already know, e.g. mid span.

Computing results in:

$$R_M^{(a)} = \tfrac{1}{2} b \times t \times (\sigma_U + \sigma_T) + \tfrac{1}{2} a \times t \times (\sigma_T + \sigma_S) = 54634.08 \text{ N}$$

$$\tau_{\text{S-L}} = -7.8 \text{ N/mm}^2$$

Note: The shear force at mid span is not significant in this case, maximum shear occurs at the supports.

d) First determine the total shear force taken by the L-section. From this, the part taken by the tube can be found. For the vertical shear force at the L-section, the shear stress distribution of the vertical part ST has to be integrated and multiplied by the thickness t. In order to do so the function of the normal stress for ST depending on the vertical position in the cross section has to be specified. Use a temporary local axis to express this function.

NOT ASKED FOR:
This computation reveals that the tube takes almost 70 % of the vertical shear.