Problem : Work & Energy (approx. 40 min)

A linear elastic beam with bending stiffness *EI* is split in to two parts which are linked with a rotational spring with spring stiffness *r*. The beam is partially loaded with a distributed load *q* as indicated in the figure. All dimensions are also specified. Axial and shear deformation are neglected.

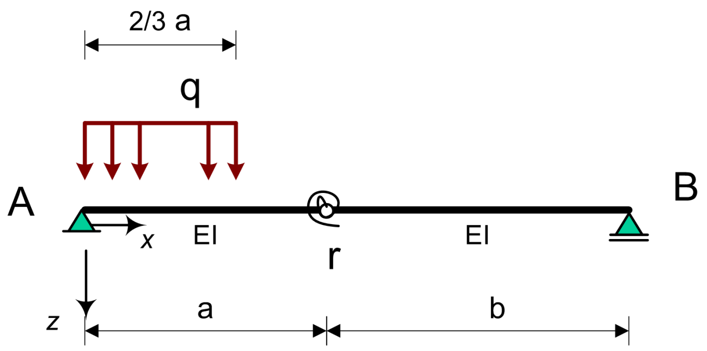


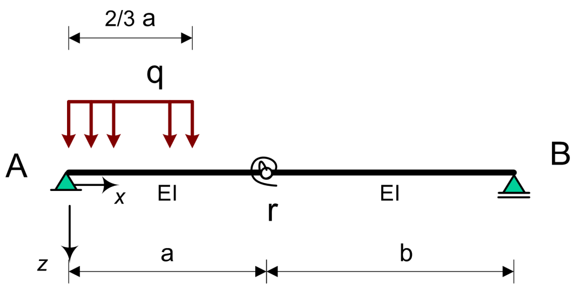
Figure 3 : Beam structure with rotational spring and (partially) distributed load.

Questions:

a) The value of the spring stiffness is of influence of the maximum moment in the structure?
b) Find the (absolute) value of the support reaction at A in kN.
c) Find the (absolute) value of the kink in the beam at the location of the spring in radians.
d) Find the amount of energy stored due to bending deformation in this structure in Joule.
e) Find the vertical displacement of the connection between the two beam parts using Castigliano's theoreme.

ANSWERS

Problem : Work & Energy (40 min)



a) This structure is statically determinate and thus the force distribution is **not** depending on the stiffness's of each of the structural parts including the rotational spring. In case the spring stiffness r becomes zero this structure is turned into a mechanism and therefore cannot be assed and is irrelevant for this question.

b) The support reactions (positive assumed both upwards) can be found based on elementary static equilibrium:

$$B_V = \frac{\frac{1}{2}q(\frac{2}{3}a)^2}{a+b}; \quad A_V = \tfrac{2}{3}qa - B_V = \frac{\frac{2}{3}qa(a+b) - \frac{1}{2}q(\frac{2}{3}a)^2}{a+b} = \frac{\frac{4}{9}qa^2 + \frac{2}{3}qab}{a+b};$$

c) The kink in the structure at the location of the spring can be found with:

$$\Delta\varphi = \frac{M_{spring}}{r} = \frac{B_V b}{r} = \frac{2qa^2 b}{9(a+b)r}$$

d) The (complementary) energy stored can be found based on the force distribution. First obtain the bending moment distribution and the moment in the rotational spring. In this case two parts have to be distinguished for the function to describe the full moment distribution:

$$M_1(x) = A_V x - \tfrac{1}{2}qx^2 \qquad 0 < x < \tfrac{2}{3}a$$
$$M_2(x) = B_V(a+b-x) \qquad \tfrac{2}{3}a < x < a+b$$
$$M_{spring} = B_V b = \frac{2qa^2 b}{9(a+b)}$$

Total energy stored due to bending using MAPLE, thus becomes:

$$E_c = \int_0^{\frac{2}{3}a} \frac{M_1^2(x)}{2EI}\mathrm{d}x + \int_{\frac{2}{3}a}^{a+b} \frac{M_2^2(x)}{2EI}\mathrm{d}x + \frac{M_{spring}^2}{2r} =$$

$$E_c = \frac{26q^2a^4\left((a^2 + \frac{48}{13}ab + \frac{45}{13}b^2)(a+b)r + \frac{135}{13}b^2 EI\right)}{10935 EI(a+b)^2 r}$$

e) In order to find the vertical displacement at the location of the rotational spring, a concentrated dummy force Q has to be added. Applying Castigliano's theorem

$$w_{spring} = \frac{\mathrm{d}E_c}{\mathrm{d}Q}$$

Finally taking $Q = 0$ will result in the requested vertical displacement only due to the distributed load. For this the support reactions and the moment distributions have to be adjusted in order to add the concentrated (dummy) load Q:

$$B_V = \frac{\frac{1}{2}q(\frac{2}{3}a)^2 + Qa}{a+b};$$

$$A_V = \frac{\frac{2}{3}qa(a+b) + Q(a+b) - \frac{1}{2}q(\frac{2}{3}a)^2 - Qa}{a+b} = \frac{\frac{4}{9}qa^2 + \frac{2}{3}qab + Qb}{a+b};$$

$$M_1(x) = A_V x - \tfrac{1}{2}qx^2 \qquad 0 < x < \tfrac{2}{3}a$$

$$M_2(x) = B_V(a+b-x) - Q(a-x) \qquad \tfrac{2}{3}a < x < a$$

$$M_3(x) = B_V(a+b-x) \qquad a < x < a+b$$

$$M_{spring} = B_V b = \frac{2qa^2b + 9Qab}{9(a+b)}$$

$$E_c = \int_0^{\frac{2}{3}a} \frac{M_1^2(x)}{2EI}\mathrm{d}x + \int_{\frac{2}{3}a}^{a} \frac{M_2^2(x)}{2EI}\mathrm{d}x + \int_a^{a+b} \frac{M_3^2(x)}{2EI}\mathrm{d}x + \frac{M_{spring}^2}{2r}$$

With MAPLE the result becomes:

$$w_{spring} = \frac{a^3bq(7a^2r + 25abr + 18b^2r + 54bEI)}{243EI(a+b)^2 r}$$

NOTE

All essential calculations are based on elementary calculus of support reactions. All other equations can be expressed in terms of these support reactions and evaluated with MAPLE. Essential prerequisite knowledge on statics is needed. The first four questions could be answered with basic knowledge on Work & Energy, the last question requires the theoretical knowledge how to apply Castigliano's theorem for this case.